SELFATOPES AND THEIR PROPERTIES

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Introduction

Through toric varieties, polytopes have been connected with projective algebraic geometry, cones and fans, ring ideals, and group actions. Polytopes are also useful in fields such as Operations Research and involve the use of combinatorics and linear algebra.

Our goal this summer was to better understand a special class of polytopes which we called a selfatope. Selfatopes first appeared in a problem in a theorem on toric varieties by Jessica Sidman and David Cox. Once we were able to recognize a selfatope, we began looking into how these polytopes might be related to each other. Along with trying to classify the examples we had created, we also tried to expand our list of examples by developing algorithms to construct more selfatopes. In the process, we were able to discover certain patterns as well as interesting, and slightly unexpected, restrictions on the existence of selfatops.

The first section of this paper will provide a general overview of terms and ideas needed in what follows. In Section 2, we will present a theorem which will outline a criterion for determining equivalence between polytopes. This section will also include a distinct class of examples. Section 3 will provide the first look into a method for creating selfatopes and the limits of this method. Section 4 will outline another method which will prove to be helpful in creating selfatopes. The last section, Section 5, contains two algorithms to generate selfatopes with some specified characteristics.

Acknowledgments

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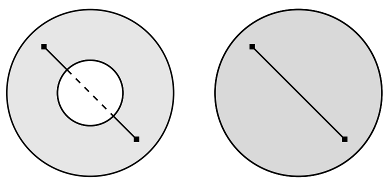
Polymake [2] for the insight it provided into the geometry of polytopes. I am very thankful for my fellow REU members; Lisa Byrne, Vince Lyzinski, Aaron Wolbach, and Frances Worek. Finally, I would like to acknowledge Professor Margaret Robinson, Mount Holyoke College, and the P-adic Analysis Group for their support throughout this summer.

1. Definitions and Background

In order to understand selfatopes, we need a basic understanding of polytopes in general. The following definitions are standard terms regarding polytopes. For more information, please refer to Ziegler or Grünbaum. [4] [3]

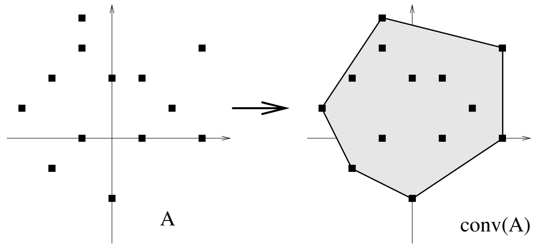
Definition 1.1. A set A is *convex* if $\forall\ p, q \in A,\ tp + (1-t)q \in A$ for $0 \leq t \leq 1$. Furthermore, the *convex hull* of a set $A \in \mathbb{R}^n$ is the intersection of all convex sets that contain A.

Example 1.2. As an example, we look at a doughnut shape and a circle. The doughnut is not convex because the line connecting the two displayed points passes outside of the doughnut. This is not true for the circle, which is in fact convex.



Definition 1.3. A *polytope* is the convex hull of a finite set of points in $\mathbb{R}^n$.

Example 1.4. In this example, A is a finite set of points in $\mathbb{R}^2$. When we take the convex hull of A, we get a convex polytope.



Definition 1.5. Let $v_1, \ldots, v_k \in \mathbb{R}^n$. Then $r_i v_i + \ldots + r_k v_k$ is a *convex combination* if all $r_i \geq 0$ and $\sum r_i = 1$.

In fact, the convex hull of a set of points $\{v_1, \ldots, v_k\}$ can be written as the convex combination of those points. That is $\text{conv}(\{v_1, \ldots, v_k\}) = \{r_1v_1 + \ldots + r_kv_k | r_i \geq 0, \sum r_i = 1\}$. For proof of this, see Ziegler. [4]

We have defined a polytope in terms of a set of points, however, we can also define it in terms of hyperplanes.

Definition 1.6. A *hyperplane* in $\mathbb{R}^n$ is the set of all solutions to a linear equation $\lambda_1 x_1 + \ldots + \lambda_n x_n = 0$ where not all $\lambda_i = 0$.

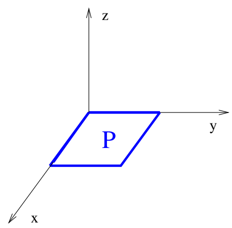
Now that we have a defined polytopes, we need to generate tools to be able to describe their properties.

Definition 1.7. An *affine hyperplane* in $\mathbb{R}^n$ is the set of all solutions to a linear equation $\lambda_1 x_1 + \ldots + \lambda_n x_n = a$ where $\lambda_i, a \in \mathbb{R}$ and not all $\lambda = 0$. If A is a subset of $\mathbb{R}^n$, the *affine hull* of A, $\text{aff}(A)$, is the intersection of all affine hyperplanes containing A.

This affine geometry allows us to define the smallest space the polytope is contained in.

Definition 1.8. The *dimension* of a polytope P is the dimension of its affine hull. A polytope is *full dimensional* in $\mathbb{R}^n$ if the dimension of P is n.

Example 1.9. The polytope P below is embedded in $\mathbb{R}^3$ but $\text{aff}(P) = \mathbb{R}^2$ which is the xy-plane. So the dimension of P is 2. Note that P is not full dimensional in $\mathbb{R}^3$.



Definition 1.10. If H is the hyperplane $a \cdot w = \alpha$ where $\alpha \in \mathbb{R}$ and $a \in \mathbb{R}^n$, then

$H^+ = \{w \in \mathbb{R}^n | a \cdot w \geq \alpha\}$

$H^- = \{w \in \mathbb{R}^n | a \cdot w \leq \alpha\}$

are the *positive* and *negative halfspaces* determined by H.

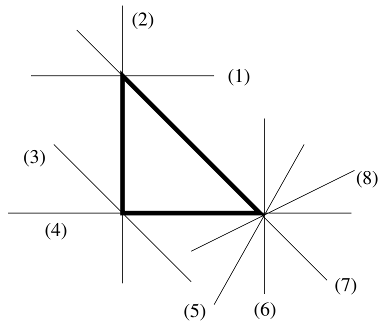
A polytope can also be described as the bounded intersection of a finite number of halfspaces, as shown in Ziegler. [4]

Definition 1.11. A hyperplane H is a *supporting hyperplane* of a polytope P if

(1) $H \cap P \neq \emptyset$

(2) P lies in H^+ or H^-.

Example 1.12. Let us look at the supporting hyperplanes of the polytope $P = \text{conv}\{(0,0),(0,1),(1,0)\}$, with thanks to Lisa Byrne for providing the following picture. Note that there are four supporting hyperplanes shown at the vertex $(1,0)$. In fact, there are infinitely many supporting hyperplanes at each vertex. They are all lines in $\mathbb{R}^2$ which intersect the vertex and keep P to one side.



(1) y = 1
(2) x = 0
(3) x + y = 0
(4) y = 0
(5) y - x = -1
(6) x = 1
(7) x + y = 1
(8) $\frac{-1}{2}$ x + y = $\frac{-1}{2}$

Not only do hyperplanes define polytopes, they also define specific aspects of polytopes.

Definition 1.13. If H is a supporting hyperplane of P, then $H \cap P$ is a *face* of P. A *vertex* is a 0-dimensional face, an *edge* is a 1-dimensional face and a *facet* is an $(n-1)$-dimensional face of an n-dimensional polytope.

We can visualize this definition using the polytope from Example 1.12. The triangle has 3 vertices at $(0,0)$, $(1,0)$, $(0,1)$ and 3 edges, the lines connecting each pair of vertices. Since the affine hull of the triangle is $\mathbb{R}^2$, the facets of this triangle are the 1-dimensional edges.

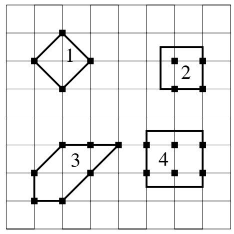
The above definitions have been true for all polytopes. However, this paper is concerned with a specific type of polytope, the selfatope. The following terms define the properties of selfatopes.

Definition 1.14. A *lattice* polytope is a polytope whose vertices all have integer coordinates.

Definition 1.15. Let P be a lattice polytope. Then P has *lattice-free edges* if each edge of P contains no lattice points other than its vertices. Furthermore, a lattice line segment has *lattice length 1* if the only lattice points on the segment are its vertices.

Example 1.16. In this example we see combinations of lattice and non-lattice polytopes with and without lattice-free edges.

The polytope #1 is a lattice polytope with lattice-free edges while polytope #2 has lattice-free edges but is not a lattice polytope. Polytope #3 is a lattice polytope, but does not have lattice-free edges, and finally polytope #4 is neither lattice nor has lattice-free edges.



Definition 1.17. Let $P \subseteq \mathbb{R}^n$ be a lattice polytope of dimension k where $k \leq n$. Let v be a vertex of P. Let w_i be the nearest lattice vectors on each edge incident to v. Then for P to be *smooth*, $\{w_i - v\}$ should form part of a $\mathbb{Z}$-basis for $\mathbb{Z}^n$ and the span of $\{w_i - v\}$ has dimension k. If $k = n$, then this means $w_i - v$ forms a basis and

$$\det \begin{pmatrix} | & & | \\ w_1 - v & \dots & w_k - v \\ | & & | \end{pmatrix} = \pm 1.$$

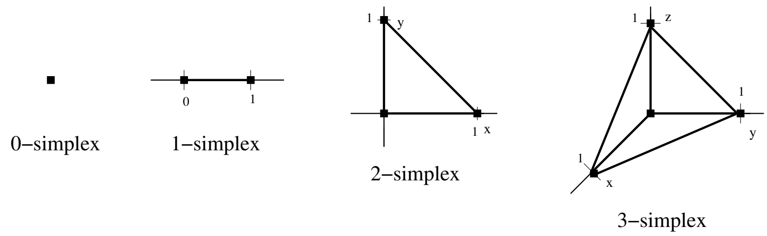
The $n \times n$ matrix formed in this process of determining smoothness for vertex v of the full dimensional polytope P will be called the *smoothness matrix* and denoted W_v.

Definition 1.18. A smooth, lattice polytope with lattice-free edges is a *selfatope*. Furthermore, a selfatope in $\mathbb{R}^2$ is a *selfagon*.

There are some standard examples of polytopes which, when sized correctly, are selfatopes as well.

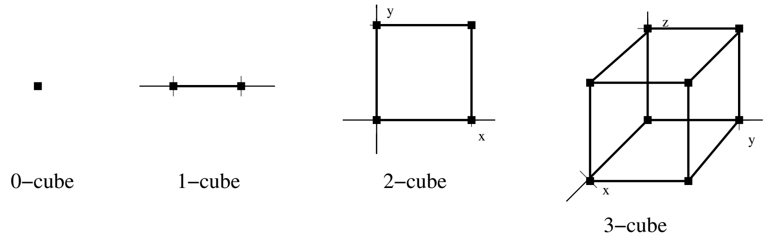
Definition 1.19. An *n-simplex* is the convex hull of $n + 1$ affinely independent points. The *standard n-simplex* is the convex hull of the standard basis vectors and $(0, \dots, 0)$: $\text{conv}\{e_1, \dots, e_n, (0, \dots, 0)\}$.

Example 1.20. The following are the standard 0,1,2, and 3-simplices.



Definition 1.21. The *standard n-cube* $\subseteq \mathbb{R}^n$ is the convex hull of all points whose coordinates are made up of 0's and 1's.

Example 1.22.



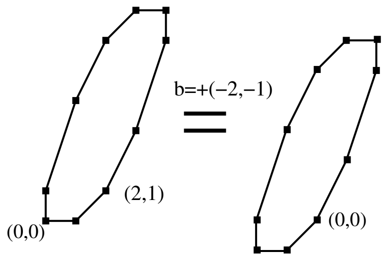
As expected, placement of a selfatope in $\mathbb{R}^n$ does not change a selfatope, as long as lattice points are preserved. That is, any selfatope will be equivalent to a congruent selfatope, no matter its placement in the hyperplane.

Definition 1.23. A *translation* of a polyope occurs when a vector b is added to every point of the polytope.

Property 1.24. If P is a selfatope in $\mathbb{R}^n$, then $P \sim P + b$ for any $b \in \mathbb{Z}^n$. In particular, $-b$ may be a vertex of P, so we see that we can translate any vertex of P to the origin.

Note that a translation by a vector $b \in \mathbb{Z}^n$ preserves all three selaftope properties. Because b is an integer vector, such a translation will act like adding the same integer onto each corresponding coordinate of all points in the selfatope. Thus, lattice vertices will translate to lattice vertices while there will still be no lattice points within each edge. Finally, the translated selfatope will still be smooth since the translation moves each lattice point an equal amount. Since this change is equal for each lattice point, the smoothness matrix will not change.

Example 1.25. If we want to move the vertex (2,1) of the 10-gon selfatope below to the origin, we must translate the entire selfagon by the vector $b = (-2, -1)$. A similar process can be carried out for any of the vertices.



Through integer translations, we begin to see how two selfatopes can be equivalent even if they are not in the same position. However, this only begins to explain equivalence of selfatopes.

2. Preserving Selfatope Properties

Now that we know what a selfatope is, we work to develop a criterion for determining equivalence between selfatopes.

Theorem 2.1. *Linear transformations that can be written as elements of $GL(n, \mathbb{Z})$ (with determinant ± 1) preserve all properties of selfatopes.*

Proof. Let $x \in \mathbb{R}^n$ denote a point in a selfatope, P. Let

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

be a matrix with elements in $\mathbb{Z}$ with determinant of ± 1.

First we must prove that the described linear transformations preserve lattice points and thus preserve the lattice vertices of the lattice polytopes. The vertices of the new polytope created by the linear transformation are found by calculating Ax for all vertices of the original polytope. We see that

$$Ax = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + \ldots + a_{1n}x_n \\ \vdots \\ a_{n1}x_1 + \ldots + a_{nn}x_n \end{pmatrix}$$

Because $a_{ij}, x_i \in \mathbb{Z}$, $\sum_{i=1}^{n} a_{ij}x_i \in \mathbb{Z}$. Thus, the vertices of the new polytope are also lattice points.

Next we must make sure the given linear transformations preserve lattice-free edges. Let us take a moment to look at A^{-1}. From linear algebra, we see that $A^{-1} = \frac{1}{\det(A)}(\mathrm{Adj}(A))$ where $\mathrm{Adj}(A)$ is the adjoint of A and $\det(A)$ is the determinant of A. Based on the definition of the adjoint, $\mathrm{Adj}(A)$ will have integer elements because A has integer elements. We defined A to have a determinant of ± 1, so $\frac{1}{\det(A)} = \pm 1$. Thus, we can see that since A has integer elements, A^{-1} will have integer elements as well.

Calculation shows $1 = \det(I) = \det(AA^{-1}) = \det(A) \cdot \det(A^{-1}) = \pm 1 \cdot \det(A^{-1})$ which implies $\det(A^{-1}) = \pm 1$. Thus, we can conclude that if A is a matrix with elements in $\mathbb{Z}$ with a determinant of ± 1, then A^{-1} is also a matrix with element is $\mathbb{Z}$ with a determinant of ± 1. Importantly, we know that A maps lattice points to lattice points because A is a matrix in $GL(n, \mathbb{Z})$, thus we can conclude that A^{-1} will also map lattice points to lattice points.

Let's assume for the moment that the original polytope P has lattice-free edges but that the image of P under A, $A(P)$, does not. Specifically, let y be a non-lattice point on an edge of P such that Ay is a lattice point on an edge of $A(P)$. But then $A^{-1}(Ay) = (A^{-1}A)y = Iy = y$. However, we just found that A^{-1} maps lattice points to lattice points so A^{-1} should map the lattice point Ay to a lattice point. But we defined y to be a non-lattice point. Thus, we have a contradiction and we find that if P has lattice-free edges, then $A(P)$ must also have lattice-free edges.

Finally, such linear transformations must preserve smoothness. Let v be a vertex of P and W_v be the smoothness matrix at v. Since we defined P to be smooth, $\det(W_v) = \pm 1$ for all v. In calculating smoothness for $A(P)$, we look at the determinant of AW_v. But, $\det(AW_v) = \det(A) \cdot \det(W_v) = \pm 1 \cdot \pm 1 = \pm 1$. Thus, the matrix A preserves smoothness.

□

Definition 2.2. A vertex of a polytope, P is in *standard position* if it lies at $(0, \ldots, 0)$ and all adjacent edges lie along the coordinate axes.

Property 2.3. Through a linear transformation and a translation, any vertex of a selfatope can be place in standard position.

This property follows directly from Theorem 2.1. Also, Theorem 2.1 allows comparison between selfatopes which appear distinct but should be considered equivalent.

Proposition 2.4. *Let P be a selfatope in $\mathbb{R}^2$. Let v be the vertex of P that is placed at the origin and let (a, b) and (c, d) be the two adjacent vertices. Since P is a selfatope, we know that $a, b, c, d \in \mathbb{Z}$ and that* $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm 1$. *The linear transformation* $\begin{pmatrix} a & c \\ b & d \end{pmatrix}^{-1}$ *will place the vertex v in standard position.*

Proof. Note that since $ad - bc = \pm 1$,

$$S = \begin{pmatrix} a & c \\ b & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \text{ or } \begin{pmatrix} -d & c \\ b & -a \end{pmatrix}.$$

Calculation shows the determinants of these matrices are both $ad - bc = \pm 1$ and $a, b, c, d \in \mathbb{Z}$ so $S \in GL(2, \mathbb{Z})$ and meets the criterion of Theorem 2.1. Also, if $ad - bc = 1$, then

$$S \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ad - bc \\ -ab + ab \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$S \cdot \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} cd - cd \\ -bc + ad \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

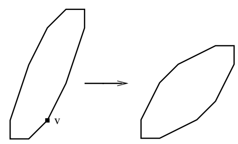
If, on the otherhand, $ad - bc = -1$, then □

$$S \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -d & c \\ b & -a \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -ad + bc \\ ab - ab \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$S \cdot \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} -d & c \\ b & -a \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} -cd + cd \\ bc - ad \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus, this matrix does in fact map v into standard position.

Example 2.5. In this example we have two 10-gons in $\mathbb{R}^2$. At first glance they seem different. However, once vertex v is moved to the origin using Property 1.24, the matrix $\begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix}$ transforms the first 10-gon into the second by Proposition 2.4. Note that it is true that $\det \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix} = 1$. Thus, we are able to see they are in fact equivalent selfatopes.



The problem with this criterion is that it is usually difficult to find the needed transformation matrix.

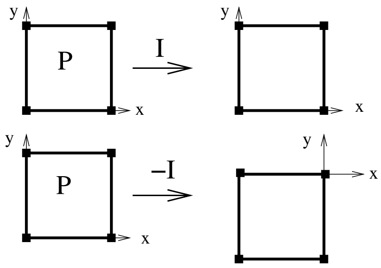
Linear Operators in $\mathbb{R}^2$

This section focuses on linear operators in the form of 2×2 matrices that preserve the three properties of selfatopes. There are several standard linear transformations in $\mathbb{R}^2$ which have a clear geometrical interpretation. [1] Selfatopes respond differently to each of these transformations. In this investigation, let P be a selfatope in $\mathbb{R}^2$.

Identity: The identity matrix, I, and the negative of the identity matrix, $-I$, that is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. We see that the determinant of these two matrices are both 1. Thus, the I and $-I$ should preserve all the properties of selfatopes.

Let $p = (p_1, p_2)$ be a point on P. Note, that $Ip = (p_1, p_2)$ and $-Ip = (-p_1, -p_2)$. As expected, applying the identity matrix to P fixes P. On the other hand, appling $-I$ to P results in a reflection about the x- and y- axis. So, as expected from the theorem, I and $-I$ preserve the properties of selfatopes.

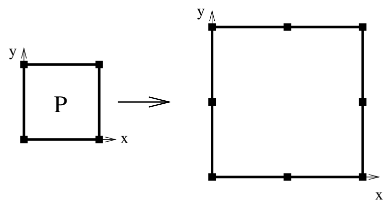
Example 2.6. Let P be the 2-cube, that is $\text{conv}\{(0,0),\ (0,1),\ (1,0),\ (1,1)\}$ Acting I and $-I$ on this selfatope, we get:



Thus, we can see that both I and $-I$ preserve selfatope properties.

Scaling: Scaling by a matrix is a way to stretch or shrink a polytope by a constant k. The general form of a scaling matrix is $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$.

Example 2.7. Let $k = 2$. Then, the scaling matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ stretches the 2-cube to the square with side length of two with vertices at $(0,0), (0,2), (2,0)$, and $(2,2)$.

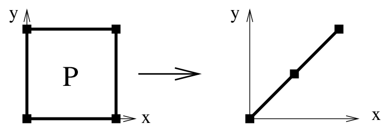


This new square does not have lattice-free edges. Moreover, $\det \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 4$.

In general, the determinant is always going to be k^2. Thus, scaling only works when $k = \pm 1$, that is, only when the scaling matrix is equal to I or $-I$.

Projections: The projections we are considering are projections of a polytope in $\mathbb{R}^2$ onto a line through the origin spanned by the unit vector (u_1, u_2). The general form of a projection matrix is $\begin{pmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{pmatrix}$. We see that, in general, the determinant of this matrix is $u_1^2u_2^2 - u_1^2u_2^2 = 0 \neq \pm 1$. However, because (u_1, u_2) is a unit vector, then it is also true that $\sqrt{u_1^2 + u_2^2} = 1$. The only integers that could meet this criteria are combinations of 0 and ± 1. So we are actually left with only the projections onto the x- and y-axes.

Example 2.8. Let P be the 2-cube and let us project P onto the line $y = x$. Thus, $(u_1, u_2) = (1, 1)$. Acting $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ on P we get:



However, this line does not have lattice-free edges.

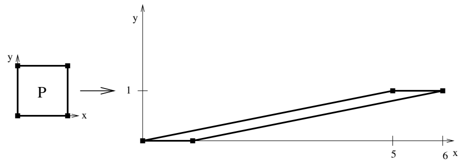
Thus, except for a few, select examples, projections onto a line in $\mathbb{R}^2$ do not preserve the properties of selfatopes.

Reflections: The reflections we are considering are reflections of a polytope in $\mathbb{R}^2$ about a line. The general form of a reflection matrix is $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ such that $a^2+b^2 = 1$. We notice that the determinant of this matrix is $-a^2 - b^2 = -(a^2 + b^2) = -1$. Thus, we might conclude that any reflection would preserve the properties of selfatopes. However, there are only a few combinations of integers such that $a^2 + b^2 = 1$. In fact, only combinations of 0 and ± 1 will meet this criteria. Thus, we see that the only reflections that actually preserve our properties are reflections about the x- and y-axes and the line $y = x$.

Rotations: We are considering rotations through an angle in the $\mathbb{R}^2$ plane. The general form of a rotation matrix is $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ such that $a^2 + b^2 = 1$. So, the determinant of this matrix is $a^2 + b^2 = 1$ by definition. However, we run into the same problem we had with reflections. Although in theory any rotation would work, our limitations of integer values for the matrix elements restricts a and b to be combinations of 0 and ± 1. So, we are only allowed rotations of multiples of $90°$. Since a rotation is the same as two reflections, so it makes sense that the limits on rotations correspond to the limitations of reflections.

Shears: There are two types of shears we will consider; horizontal and vertical. Shearing can be pictured as putting your hand on a side of a big block of Jello and pushing parallel to that side. The general form of a horizontal shear is $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ and of a vertical shear it is $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$.

Example 2.9. We can shear the 2-cube with vertices at $(0,0)$, $(0,1)$, $(1,0)$, and $(1,1)$ by the matrix $\begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}$. With this shear, we will get a parallelogram with vertices at $(0,0),(1,0),(5,1)$ and $(6,1)$ respectively.



A calculation shows that this is in fact a selfatope.

In general, the determinant of a shear matrix is 1. Thus, shears do in fact preserve the properties of selfatopes.

We can conclude that not all linear operators preserve the properties of selfatopes. While all linear operators that are elements of $GL(2,\mathbb{Z})$ preserve properties of selfatopes, some matrices in $GL(2,\mathbb{Z})$ can be classifed as specific geometric operations. We have considered some linear operations and found that rotations, reflections, and shears preserve the properties of interest. However, there are only a few, select rotations and reflections which meet the requirements for our specified matrices. Scalings and projections, on the other hand, do not preserve the properties of selfatopes.

Of course, linear transformations do not create new selfatopes, but simply change the appearance of those we know. There are, however, ways to generate new selfatopes from existing ones.

3. Pyramiding of Selfatopes

Pyramiding is a method of creating polytopes in $\mathbb{R}^n$ from polytopes in $\mathbb{R}^{n-1}$. Unlike linear transformations, pyramiding creates polytopes that are not equivalent.

Definition 3.1. If P is a d-dimensional polytope in $\mathbb{R}^n$ where $n > d$, then the *pyramid* over P is $pyr(P) \equiv \text{conv}(P, p)$, where $p \notin \text{aff}(P)$. The pyramid has dimension $d+1$.

In order to investigate pyramids in relation to selfatopes, smoothness will be considered in $\mathbb{R}^d$ and $\mathbb{R}^{d+1}$. The new point, p, must be placed on a lattice point, otherwise it will be impossible for the new polytope to have vertices on lattice points.

First, we will clarify how pyramids are to be created. We start with a polytope, P, of dimension d in $\mathbb{R}^d$. To create a pyramid of P, we embed P in the $x_{d+1} = 0$ hyperplane of $\mathbb{R}^{d+1}$. For example, the point $x = (x_1, \ldots, x_d) \in P$ is placed at $(x_1, \ldots, x_d, 0)$ in $\mathbb{R}^{d+1}$. Then, the new point, p, is placed in $\mathbb{R}^{d+1}$ such that for p, $x_{d+1} \neq 0$.

Theorem 3.2. *All selfatope pyramids must come from selfatopes and are in fact simplices.*

Proof. The idea of this proof is to show that selfatope pyramids cannot be smooth, lattice polytopes with lattice-free edges unless the original polytope was also a smooth, lattice polytope with lattice-free edges.

Suppose that a pyramid is created from a non-lattice polytope. When a pyramid is constructed, the original vertices remain in the new polytope. Thus, the new polytope created in pyramiding will continue to have the non-lattice point vertices and so will not be a lattice polytope. Therefore, in order to create a lattice polytope in pyramiding, the original polytope must be a lattice polytope.

Now suppose that a pyramid is created from a polytope without lattice-free edges. When a new pyramid is built, the original edges are incorporated into the new, higher dimensional polytope. Thus, the pyramid will not have lattice-free edges. Therefore, to build a polytope with lattice-free edges by pyramiding, the original polytope must also have lattice-free edges.

Finally, suppose that a pyramid is created from a polytope in $\mathbb{R}^d$ that has at least one vertex that is not smooth, say vertex $v = (v_1, \ldots, v_d)$. Let $w_1, \ldots, w_d$ be the nearest lattice vectors on edges incident to v_i. We check for smoothness by finding the determinant of

$$M = \begin{pmatrix} w_{11} - v_1 & \ldots & w_{n1} - v_1 \\ \vdots & & \vdots \\ w_{1n} - v_n & \ldots & w_{nn} - v_n \end{pmatrix}.$$

Since we are assuming P is not smooth at v, then $\det(M) \neq \pm 1$. Next, to form the pyramid the original polytope is placed in the $x_{n+1} = 0$ hyperplane of $\mathbb{R}^{n+1}$ and a point, p, is added anywhere in $\mathbb{R}^{n+1}$, except the 0 hyperplane, say at $(p_1, \ldots, p_n, p_{n+1})$ where $p_{n+1} \neq 0$. The new vertex has the following for it's "smooth" matrix:

$$S = \begin{pmatrix} w_{11} - v_1 & \ldots & w_{n1} - v_1 & p_1 - v_1 \\ \vdots & & \vdots & \vdots \\ w_{1n} - v_n & \ldots & w_{nn} - v_n & p_n - v_n \\ 0 & \ldots & 0 & p_{n+1} \end{pmatrix}.$$

But then, expanding along the bottom row shows that $\det(S) = \pm \det(M) \cdot p_{n+1}$. However, if $\det(M) \neq \pm 1$, then for $\det(S)$ to equal ± 1, $p_{n+1} =$

$\frac{1}{\det(M)}$. However, p_{n+1} must be a lattice point in order for the pyramid to be a lattice polytope, and this is only possible if $\det(M) = \pm 1$. Thus we have a contradiction and it is true that for the pyramid to be smooth, the original polyope must be smooth.

Note that this computation also shows that, for the pyramid to be smooth, p_{n+1} must equal ± 1 if $\det(S) = \pm 1$. Thus, when creating a pyramid from a polytope, the new point, p must be placed in the hyperplane one above or one below the $x_{n+1} = 0$ hyperplane of $\mathbb{R}^{n+1}$.

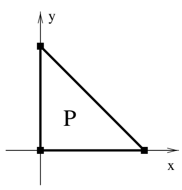
Now that we know that pyramiding a selfatope returns a selfatope, it remains to prove that this process only works when the origianl polytope is a simplex.

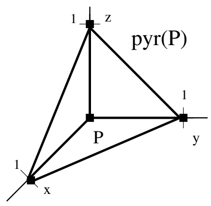
Definition 3.3. Let a set of n points in $\mathbb{R}^n$ be in *linearly general position* if every k of them affinely span a $k-1$ hyperplane.

For example, if four points are in linearly general position, any three of those points span a 2-plane. By definition of smoothness, it is not possible for a polytope in $\mathbb{R}^n$ to have a vertex with more than n edges incident to it. So, for a pyramid to be smooth at p, p must have $\leq n$ adjacent edges which implies $pyr(P, p)$ has $\leq n+1$ vertices. However, for P to be an n-dimensional polytope, it must have $\geq n+1$ vertices. If P had less than $n+1$ vertices, they would lie in a $n-1$ hyperplane and then P would not be n-dimensional. Thus, if we have a polytope that is full dimensional in $\mathbb{R}^n$ and is smooth, it has exactly $n+1$ vertices. However, by definition, a polytope in $\mathbb{R}^n$ with $n+1$ vertices is equivalent to the n-simplex. Thus, every selfatope pyramid is a simplex.

□

Example 3.4. Let P be a full dimensional selfatope in $\mathbb{R}^2$ over which we want to pyramid. By Theorem 3.2, P must be the 2-simplex, that is, a triangle. Next we place P into the $z = 0$-hyperplane and add in a point p. By Theorem 3.2, we know that p must be placed in the $z = \pm 1$-hyperplane. But, then $pyr(P)$ is simply a selfatope in $\mathbb{R}^3$ with 4 vertices. By Definition 1.19, we know this to be the 3-simplex.





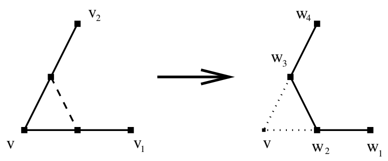
Although pyramiding, gave us an interesting way to create selaftopes, the simplices generated are not truly new selfatopes for us to study.

4. The Chopping Method

The 3P Chop Method of constructing selfatopes from selfatopes was first observed by David Cox. The Chopping Method provides a new way to create selfatopes, provided certain criterion are met.

Definition 4.1. Let $P \subseteq \mathbb{R}^n$ be a full dimensional selfatope. With the *Chopping Method*, an n-simplex is cut off at a vertex, v, of P. The new polytope is $Chop(P)$.

The Chopping Method is easily visualized in $\mathbb{R}^2$. Since P is a full dimensional selfatope, then every vertex, v has two edges adjacent to it. If a selfatope scaled by 2 or greater, then between every set of vertices, there is at least one lattice point on each edge. A 2-simplex is simply a triangle so cutting off a 2-simplex reduces to connecting the lattice points closest to v along the adjacent edges.



Thus, with the Chopping Method, the dotted edge is added while the vertex v is cut off.

While the properties of the polytope P determine whether or not $Chop(P)$ is a selfatope, there is a clear property regarding smoothness of $Chop(P)$.

Lemma 4.2. *Let $P \subseteq \mathbb{R}^n$ be a lattice polytope and let mP be P scaled by a factor of m where $m \geq 2$ and $m \in \mathbb{Z}$. Let v be a vertex of P. If P is smooth at v, then the vertices constructed by cutting off an n-simplex at v are also smooth.*

Proof. We will look at $mP \subseteq \mathbb{R}^n$, a lattice polytope with vertex v smooth. Using a translation, Property 1.24 implies we can move v to the origin. Because there are n adjacent edges to v, and $P \subseteq \mathbb{R}^n$, we can transform all the edges into standard position by Property 2.3. A set of $n-1$ edges will lie in the $x_n = 0$ hyperplane with the last edge not in the span of this plane. Without loss of generality we let w be the first lattice point away from v on this n-th edge. Keeping this coordinate system in mind we can write the smoothness matrix of v as Id_n, the $n \times n$ identity matrix, which has a determinant of 1.

Now let us look at any one of the vertices created by cutting mP at v, say v_1. Ths smoothness matrix of mP at v_1 is

$$\begin{pmatrix} 2-1 & 0 & 0 & \dots & 0 \\ -1 & 1 & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ -1 & 0 & \dots & 1 & 0 \\ -1 & 0 & \dots & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ -1 & 1 & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ -1 & 0 & \dots & 1 & 0 \\ -1 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

But note that this matrix has a determinant of 1. Thus, v_1 is a smooth vertex. Since v_1 was arbitrary, this process can be repeated for any of the new vertices. Thus, every new vertex of the polytope formed by cutting a simplex off of each vertex of mP is smooth and therefore the new polytope is smooth.

□

5. Creating Selfatopes Using the Chopping Method

Chopping smooth vertices of certain polytopes preserves smoothness. However, the Chopping Method, when joined with scaling, can also be used to create selfatopes from selfatopes.

Algorithm 5.1. **<u>The 3P Chop Method</u>**

(1) Begin with a selfatope, P.
(2) Scale P up by a factor of 3, creating $3P$.
(3) Perform the Chopping Method on each of the vertices of $3P$, creating $Chop(3P)$.

The 3P Chop Method is in fact a handy method to create selfatopes.

Theorem 5.2. *When the 3P Chop Method is performed on a selfatope, P, then $Chop(3P)$ will also be a selfatope.*

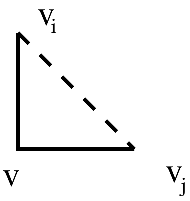
Proof. Let v be a vertex of $3P$ with v_i the nearest lattice point on some edge adjacent to v. Thus, v_{ij} is the j-th component of the nearest lattice point on the i-th edge adjacent to v.

For $Chop(3P)$ to be a selfatope, it must be a lattice polytope with lattice-free edges and be smooth at each vertex.

By construction, $Chop(3P)$ is formed by making cuts at lattice points of $3P$. Thus, $Chop(3P)$ will be a lattice polytope.

Next, we look to find if $Chop(3P)$ has lattice-free edges. In this construction, there are two types of edges; first the edges created by the cuts, and second the edges of $3P$ left by the cuts. For each pair

v_i, v_j, we have:



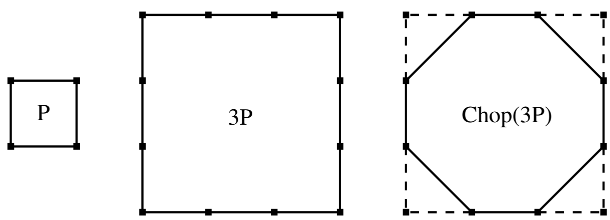
where the solid edges are of lattice length one. In this process we delete the solid edges and add the dotted edge. We know by Property 2.3 that any two vectors of lattice length one can be transformed into the standard position at the origin. This implies that the edge connecting v_i to v_j must have lattice length one as well. Thus, the new edges created by the cuts are lattice free. Now we only have to look at the edges of $3P$ left by the cuts. These edges are created from an edge of lattice length three and then a cut is made at each vertex. These cuts remove a section of lattice length one from the edge. Because each edge has two vertices, this means each original edge loses two lattice length sections. Since we started with lattice length three, the edge remaining must have lattice length one. Thus, for any type of edge of C, the edges are lattice-free.

To see that $Chop(3P)$ is smooth at each vertex, we apply Lemma 4.2 at each vertex of $Chop(3P)$. Thus, we see that $Chop(3P)$ is in fact smooth.

□

Let us look at an example of the 3P Chop to create a selfatope.

Example 5.3. Let P be the standard square in $\mathbb{R}^2$. Then $3P$ is simply the square scaled up by 3 and $Chop(3P)$ is the 8-gon selfatope.



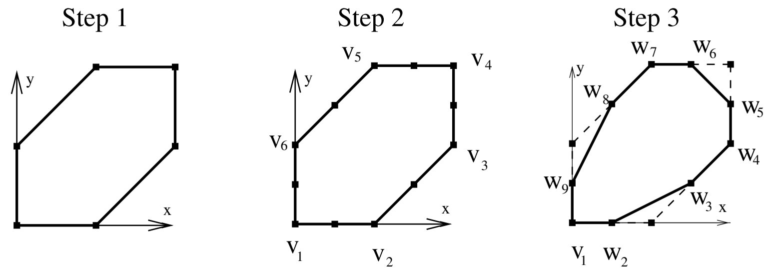
The Chopping Method also provides a way to create any selfagon with $3n$ sides from a selfagon with $2n$ sides.

Algorithm 5.4. **The 2P Chop Method**

(1) Start with an $2n$-gon selfatope $P \in \mathbb{R}^2$, where P has dimension two.
(2) Scale P up by 2 to generate $2P$.

(3) Perform the Chop Method on half of the vertices of $2P$, that is on every other vertex.

As an example, we will let $n = 3$ so P is the six-gon in $\mathbb{R}^2$.



So, from the original six-gon, we cut off vertices v_2, v_4, and v_6. With this process, we have now created a selfatope with $3n$ sides. In our example, we have created the 9-gon. Of course, this method can be used to create a multitude of selfatopes.

Theorem 5.5. *In $\mathbb{R}^2$, if we start with a $2n$ sided selfatope P, the 2P Chop Method from Algorithm 5.4 creates a $3n$ sided selfatope, Q.*

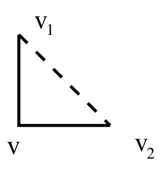
Proof. Let v be a vertex of $2P$. Let f_1 and f_2 be the two edges adjacent to v with v_1 and v_2 the closest lattice points to v along these edges, respectively.

For Q to be a selfatope, Q must be lattice, have lattice-free edges, and be smooth at each vertex.

First we will see if Q is a lattice polytope. By construction, Q is formed by making cuts at half of the vertices of $2P$. The cuts made to form Q are made by connecting v_1 and v_2, for all pairs of edges adjacent to half of the $v \in 2P$. Thus, the new vertices of Q are now at v_1 and v_2. Since v_1 and v_2 were defined to be lattice points, this implies that Q is a lattice polytope.

Now we will find if Q has lattice-free edges. In this construction, there are two types of edges; first the edges created by the cuts, and second, the edges of $2P$ left by the cuts.

First, for the pair of lattice points, v_1, v_2, we have:



where the solid edges are of lattice length one. In this process we delete the solid edges and add the dotted edge. By Property 2.3, any two vectors of lattice length one can be transformed into the standard position at the origin. This implies that the edge connecting v_1 and v_2 must have lattice length one as well. Thus, the new edges created by the cuts are lattice-free.

Now we only have to look at the edges of $2P$ left by the cuts. These edges are created from an edge of lattice length two and then a cut is made at half of the vertices. These cuts remove one lattice length section of the edge. Because each edge has two vertices, this means each original edge loses one lattice length section. Since we started with lattice length two, the edge remaining must have lattice length one. Thus, for any type of edge of Q, the edges are lattice-free.

To see that Q is smooth, we first note that in Q, there are two types of vertices. First there are the vertices of $2P$ which were not cut off. Since we started with a selfatope, Q is smooth at these vertices. The other type of vertices are those created by the cuts in the algorithm. By Lemma 4.2, we know these vertices are also smooth. Thus, in $\mathbb{R}^2$, any polytope created with this method will be smooth.

Now we know that Q is in fact a selfatope, it is left to prove that Q has $3n$ sides. Because we are working in $\mathbb{R}^2$, there are an equal number of edges and vertices of P and $2P$, namely $2n$ of each. In the algorithm, half of the vertices, n of them, are cut off and two more vertices are created in place of each vertex. So, n vertices of the original $2n$ vertices are cut off and $2n$ are added on. This implies there are $2n-n+2n = 3n$ vertices in Q. But, we are still in $\mathbb{R}^2$ so Q must also have $3n$ sides. Therefore, the selfatope created by this algorithm will have $3n$ sides given the original selfatope had $2n$ sides.

□

Thus, with these two algorithms, we are now able to create any selfagon with $2n$ and $3n$ sides. It still remains to find an algorithm to create other selfagons as well as selfatopes in any dimension.

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