Selfatopes and Their Properties

A Look into Polytopes

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Summer 2005

Why do we care about Selfatopes?

Problem: In a theorem by Jessica Sidman and David Cox dealing with toric varieties, specific polytopes kept appearing. These polytopes are what we now call selfatopes.

Goal: To find and understand selfatopes, with a view towards classification.

What is a Selfatope?

Definition: A *polytope* is the convex hull of a finite set of points.

Definition: The *convex hull* of a set $A \in \mathbb{R}^n$ is the intersection of all convex sets that contain A, conv(A).

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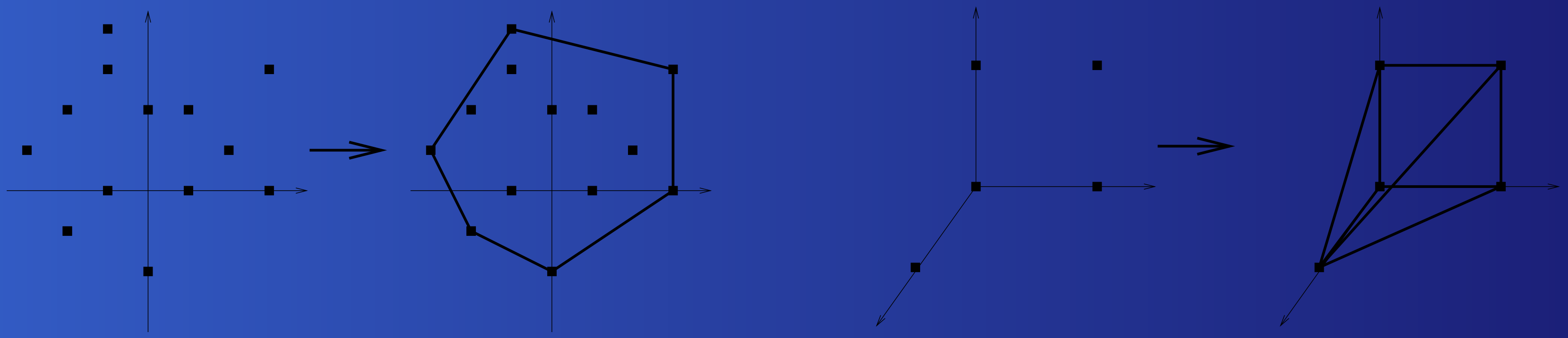

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v: vertex

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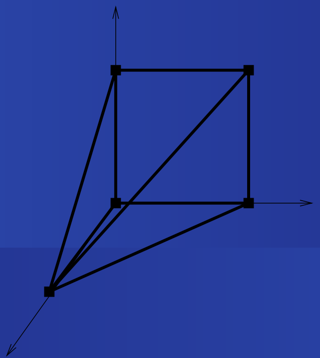

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v: vertex

w_i: nearest lattice point along edge i off of v.

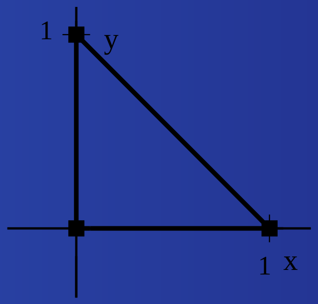
Lattice-free edges: No points with integer coordinates along any edge, except for at vertices.

Simplex

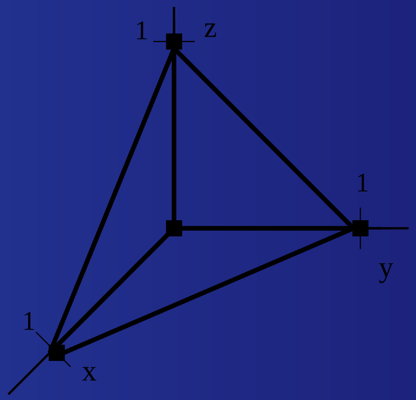
The *standard n-simplex* is the convex hull of the standard basis vectors and $(0, \ldots, 0)$: conv$\{e_1, \ldots, e_n, (0, \ldots, 0)\}$.

0−simplex

1−simplex



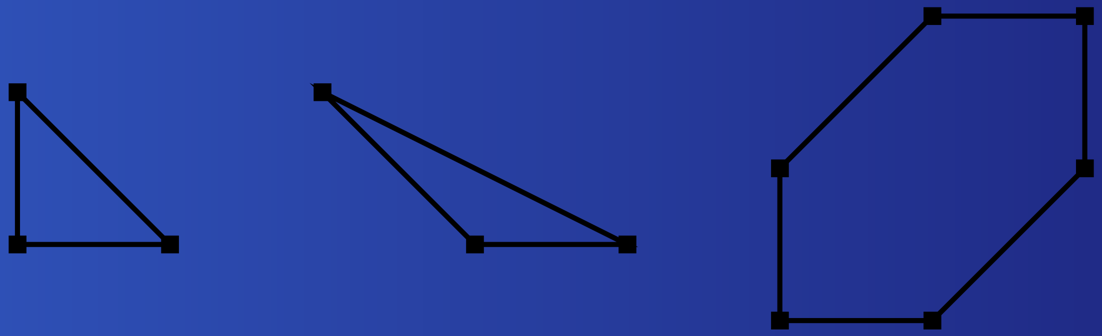
2−simplex



3−simplex

Examples of Selfatopes

Selfatopes



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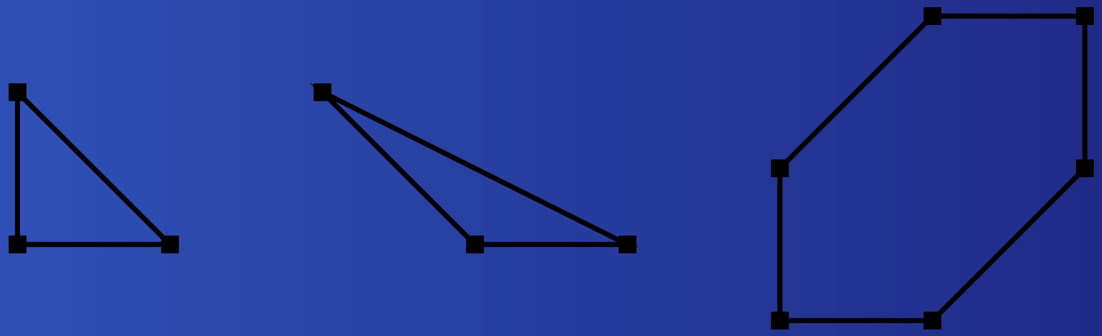
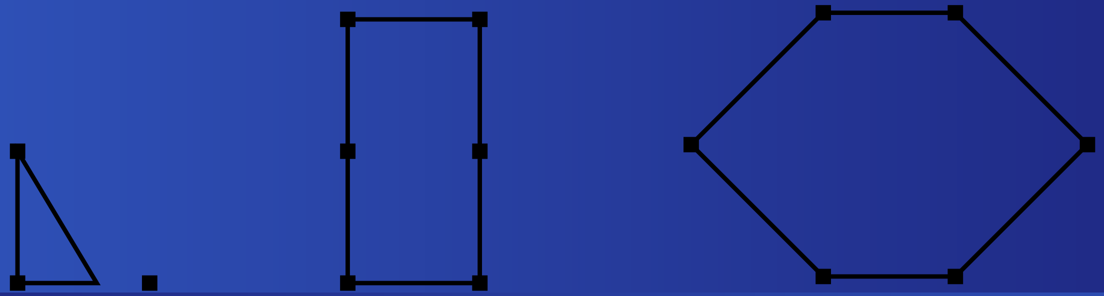


Not Selfatopes



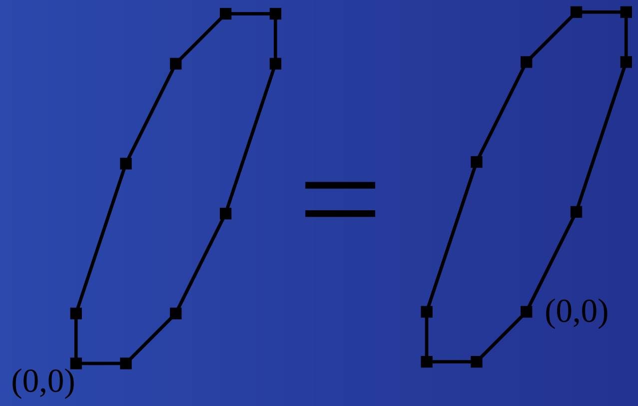
The Question of Equivalence

Equivalences

Two types of equivalence:

(1) Translation (If lattice points go to lattice points)

Example:



Equivalences

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(2) Linear Transformation

Theorem: Linear transformations that can be written as elements of $GL(n, \mathbb{Z})$ with determinant ± 1 preserve all properties of selfatopes.

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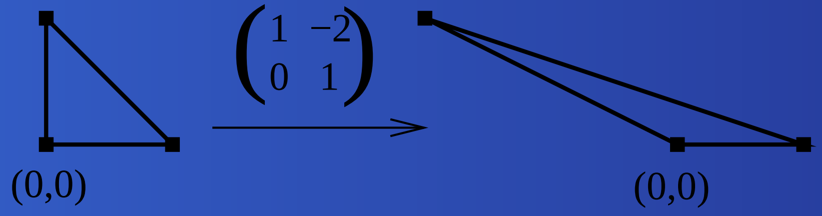
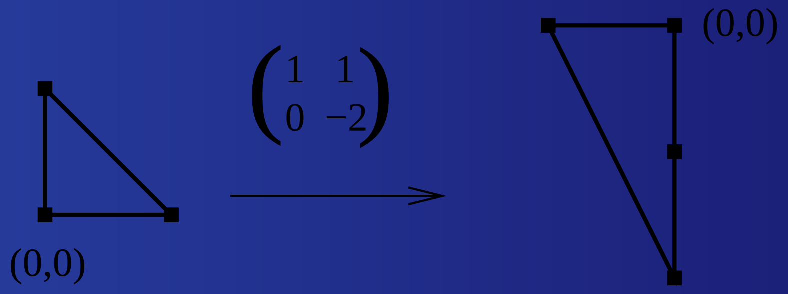
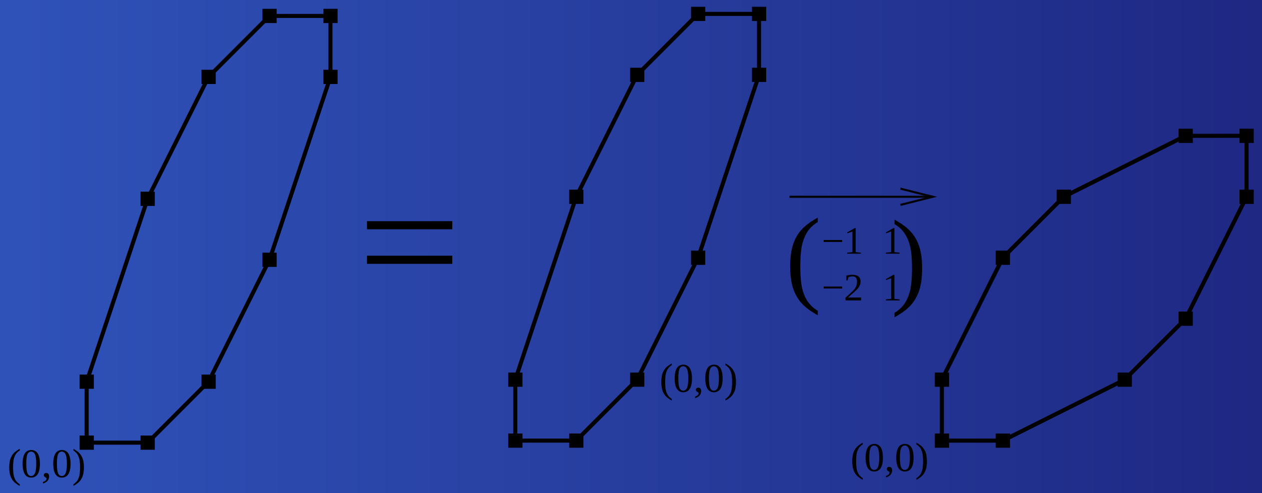




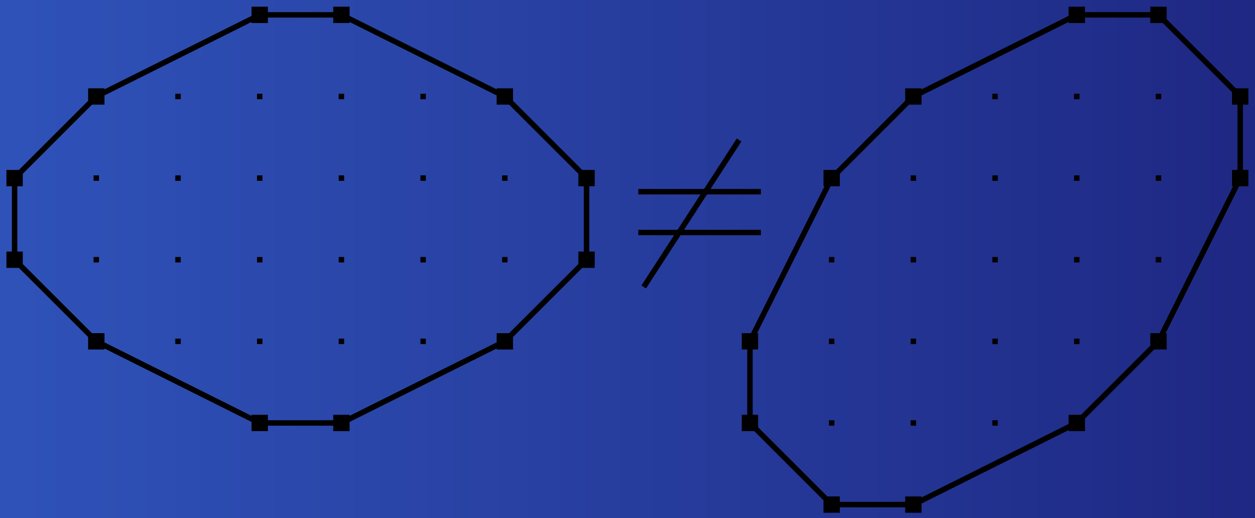
Equivalences

Translations and Linear Transformations



Equivalences

Limits



Equivalence in $\mathbb{R}^2$

Linear Transformations in $\mathbb{R}^2$:

Rotations:

General Form: $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}, a^2 + b^2 = 1$

First thought: Any rotation will work.

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Linear Transformations in $\mathbb{R}^2$:

<u>Rotations</u>:

General Form: $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, $a^2 + b^2 = 1$

Reality: $a, b = 0, \pm 1 \implies$ rotations of 90°.

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Linear Transformations in $\mathbb{R}^2$:

Reflections:

General Form: $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ such that $a^2 + b^2 = 1$

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Linear Transformations in $\mathbb{R}^2$:

Reflections:

General Form: $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ such that $a^2 + b^2 = 1$

Reality: As with rotations: $a, b = 0, \pm 1 \implies$ reflections about the x- and y-axes.

Equivalence in $\mathbb{R}^2$

Linear Transformations in $\mathbb{R}^2$:

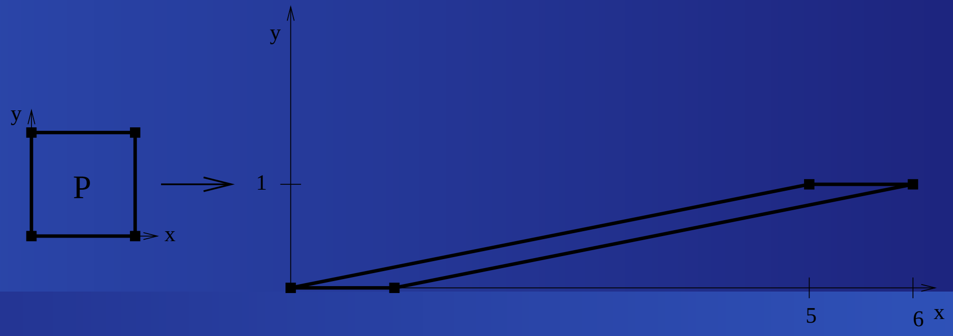
Shears:

General Form:

Horizontal: $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ Vertical: $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$.

Example:

$$\begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}$$



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<u>Shears</u>:
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Horizontal: $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ Vertical: $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$.

Reality: All shears preserve selfatope properties.

How Do We Create New Selfatopes?

The 3P Chop Method

(1) Begin with a polytope $P \in \mathbb{R}^n$

(2) Scale P up by 3 = $3P$

(3) Cut an n-simplex off at each vertex of $3P$ = $Chop(3P)$

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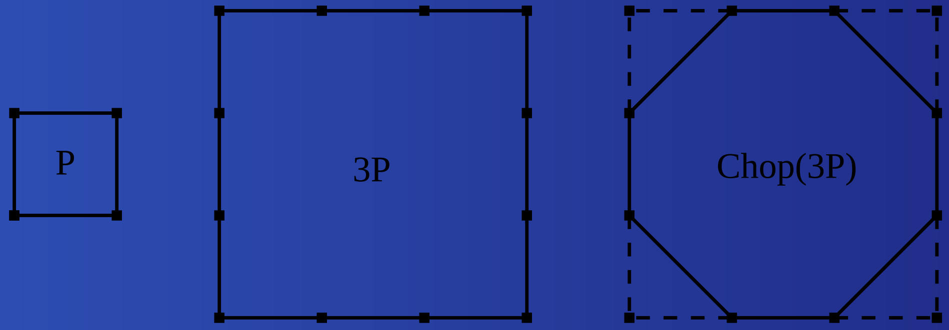

The 3P Chop Method

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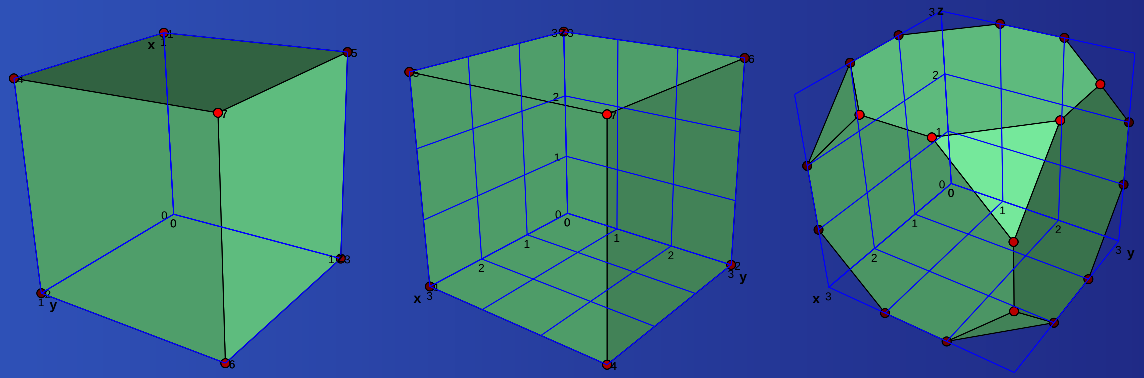

The 3P Chop Method

Properties:

In $\mathbb{R}^2$: creates $2n$-gons from n-gons

Question: How to create selfagons in $\mathbb{R}^2$ that have an odd number of sides?

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In $\mathbb{R}^2$: creates $2n$-gons from n-gons

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Partial Answer: The 2P Chop Method

The 2P Chop Method

Limits:

Works only in $\mathbb{R}^2$

Begin with $2n$-gon

Capabilities:

Build any $3n$-gon

The 2P Chop Method

(1) Begin with an $2n$-gon selfatope $P \in \mathbb{R}^2$
(2) Scale P up by 2 to generate $2P$
(3) Cut a 2-simplex off of half of the vertices of $2P$

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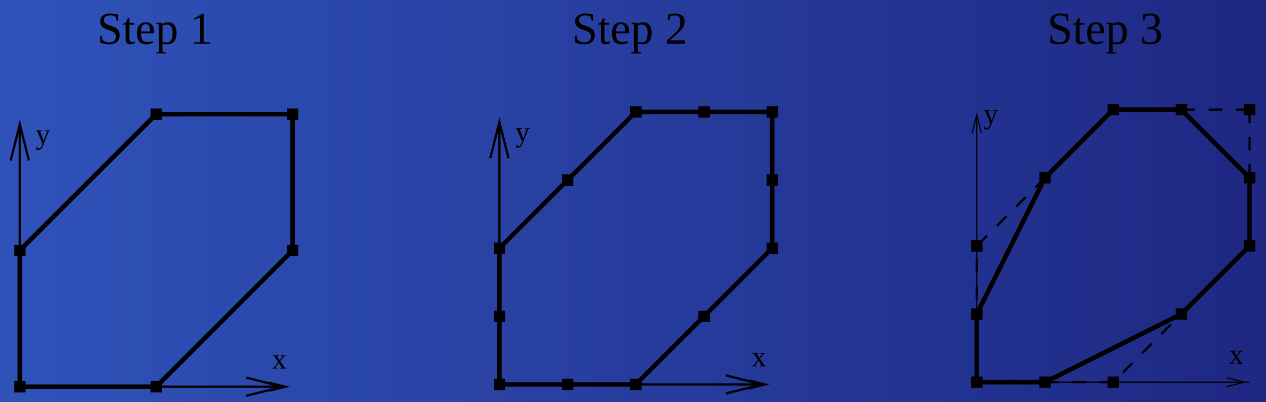
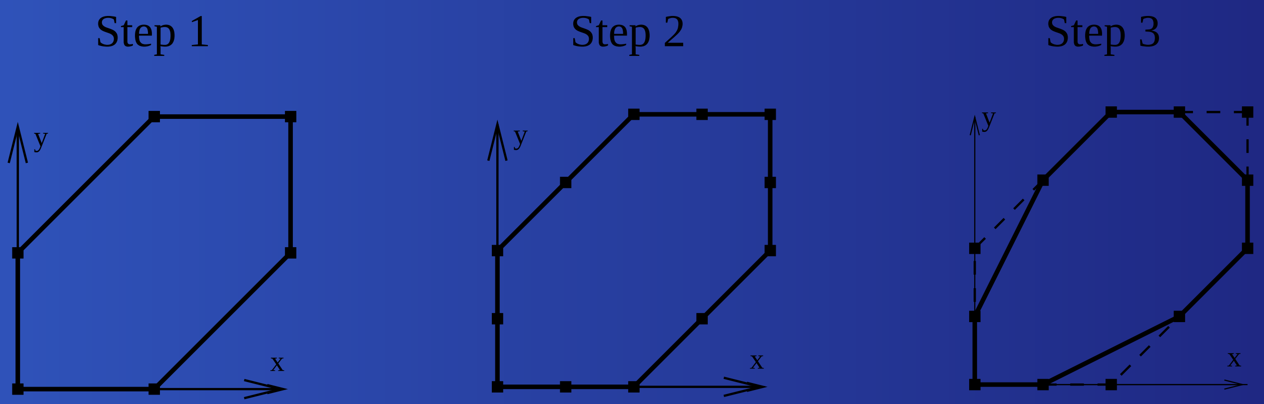


The 2P Chop Method

Theorem: In $\mathbb{R}^2$, if we start with a $2n$ sided selfatope P, the 2P Chop Method creates a $3n$ sided selfatope.
Example: $n = 3 \implies$ 6-gon$\rightarrow$ 9-gon.



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Classification:

1. Translations
2. Linear Operations

Creation: The 3P Chop and 2P Chop Methods

Conclusion

Open Questions

Determining equivalence can be tedious

3P Chop and 2P Chop Methods do not create all selfatopes

Acknowledgements

- Jessica Sidman for her guidance throughout this summer
- David Cox for the original ideas behind the 3P Chop Method
- My fellow REU students; Lisa Byrne, Vince Lyzinski, Aaron Wolbach, and Frances Worek
- Polymake for insight into the geometry of polytopes
- Funding for the work presented here provided by the NSF Grant #DMS-0353700.