SELFAGONS IN THE PLANE

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Introduction

The theory of convex polytopes dates back to the ancient Greeks and yet still offers insights into the cutting edge fields of mathematics today. For example, through toric varieties, polytopes play a crucial role in projective algebraic geometry. Polytopes have also proven themselves useful in fields as far ranging as combinatorics and operations research.

The polytopes that our group has chosen to study this summer are a special class of polytopes, known as selfatopes. Selfatopes are lattice polytopes that are smooth and have lattice free edges. We studied the various underlying properties of these polytopes in relation to each other and in relation to their smooth toric varieties. We arrived at various notions of equivalence of these selfatopes, and generated examples of selfatopes in the plane as well as in higher dimensions. We also studied the interesting restrictions that limited the possibilities of existence.

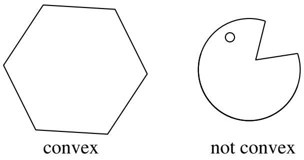
This paper's main focus is on selfatopes in the plane, with a particular emphasis on the various selfagons that cannot exist because of the rigidity of the selfatope conditions. Section 1 provides a general background into polytope theory and defines all the basics that would be needed to understand Section 2, which gives concrete proofs for the impossibility of two particular selfatopes in the plane. Section 3 gives an alternate proof of part of Section 3 using the much more elegant method of inner normal fans, which offers a deeper insight into the reasons behind why certain selfatopes cannot exist.

1. Definitions and Background Material

These are a few standard polytope definitions, followed by a few definitions that help to classify our special class of polytopes, the selfatopes. For further resources and background material refer to the books by either Ewald [1] or Ziegler [5].

Definition 1.1. A set A is *convex* if $\forall\, p, q \in A,\ tp + (1-t)q \in A$ for $0 \le t \le 1$.

Example 1.2.

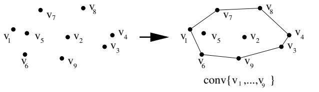


Lemma 1.3. *An intersection of convex sets is convex.*

Proof. Let C_i be convex sets, $i \in I$. Let $C = \bigcap_{i \in I} C_i$. If $p, q \in C$ then $p, q \in C_i\ \forall i$. Therefore by convexity of C_i, $tp + (1-t)q \in C_i\ \forall i,\ 0 \le t \le 1$ and thus $tp + (1-t)q \in C,\ 0 \le t \le 1$. ☐

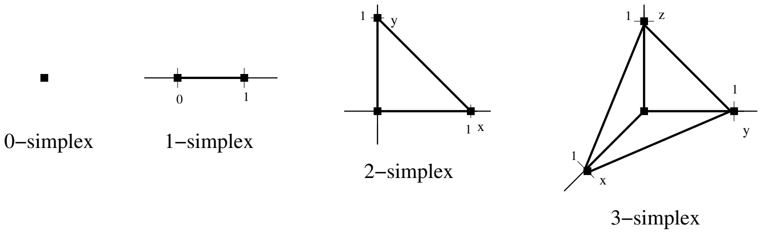
Definition 1.4. The *convex hull* of a set $A \in \mathbb{R}^n$ can be defined as the intersection of all convex sets that contain A.

Example 1.5.



Definition 1.6. A *polytope* is the convex hull of a finite set of points in $\mathbb{R}^n$.

Example 1.7. The *standard n-simplices* are polytopes that are defined as the convex hulls of the standard basis vectors and the origin in $\mathbb{R}^n$; $n - simplex = \mathrm{conv}\{e_1, \dots, e_n, (0, \dots, 0)\}$.



As its name would imply, a convex hull of a set is convex by definition. Therefore since polytopes are the convex hulls of a finite set of points, all polytopes are convex.

Definition 1.8. A *lattice polytope* is a polytope whose vertices all have integer coordinates.

Example 1.9. $P = conv\{(0,1,1),(1,0,0),(0,0,0),(1,1,1)\}$ is a lattice polytope $\in \mathbb{R}^3$.

Definition 1.10. A *hyperplane* in $\mathbb{R}^n$ is the set of all solutions to a linear equation $\lambda_1 x_1 + \ldots + \lambda_n x_n = 0$ where not all $\lambda_i = 0$. Note that $\vec{0}$ is in every hyperplane as $\vec{0} \cdot \vec{x} = 0$ for every $x \in \mathbb{R}^n$.

Definition 1.11. An *affine hyperplane* in $\mathbb{R}^n$ is the set of all solutions to a linear equation $\lambda_1 x_1 + \ldots + \lambda_n x_n = a$ where not all $\lambda_i = 0$. Note that $\vec{0}$ is not necessarily in every affine hyperplane.

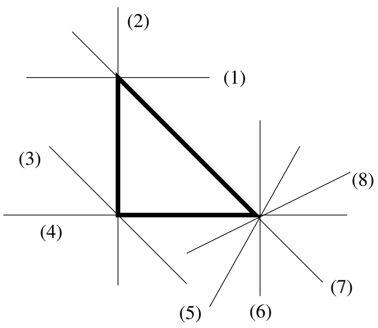
Definition 1.12. Let A be a subset of $\mathbb{R}^n$. The *affine hull* of A, $\text{aff}(A)$, is the intersection of all affine hyperplanes containing A.

Definition 1.13. The *dimension* of a polytope P is the dimension of its affine hull. A polytope is *full dimensional* in $\mathbb{R}^n$ if the dimension of P is n.

Definition 1.14. If H is a hyperplane, then
$H^+ = \{w \in \mathbb{R}^n | a \cdot w \geq \alpha\}$
$H^- = \{w \in \mathbb{R}^n | a \cdot w \leq \alpha\}$
are the *positive* and *negative halfspaces* determined by H.

Definition 1.15. A hyperplane H is a *supporting hyperplane* of a polytope P if
(1) $H \cap P \neq \emptyset$
(2) P lies in H^+ or H^-.

Example 1.16. Let $P = conv\{(0,0),(0,1),(1,0)\}$ be a polytope $\in \mathbb{R}^2$.

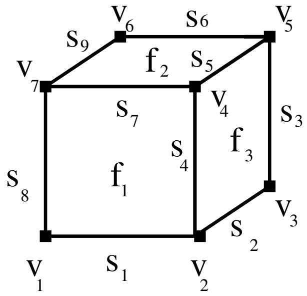


The supporting hyperplanes of P are:

1. $y = 1$	5. $y - x = -1$
2. $x = 0$	6. $x = 1$
3. $x + y = 0$	7. $x + y = 1$
4. $y = 0$	8. $\frac{-1}{2}x + y = \frac{-1}{2}$

Definition 1.17. If H is a supporting hyperplane of P, then $H \cap P$ is a *face* of P. 0-dimensional faces are *vertices*, 1-dimensional faces are *edges* and the $n-1$-dimensional face of an n-dimensional polytope is a *facet.*

Example 1.18. Let P be the polytope shown below.



The facets of P are labelled $f_1, f_2, \ldots$, the edges of P are labelled $s_1, s_2, \ldots$ and the vertices of P are labelled $v_1, v_2, \ldots$.

The following terms define certain characteristics of the polytopes we are concerned with.

Definition 1.19. Let P be a lattice polytope. The polytope P has *lattice-free edges* if the only lattice points on the edges of P are the vertices. Furthermore, a lattice line segment has *lattice length 1* if the only lattice points on the segment are its vertices.

Note that by construction, lattice polytopes with lattice free edges have edges of lattice length 1.

Definition 1.20. Let $P \in \mathbb{R}^n$ be a lattice polytope. Then P is *smooth* if for each vertex $v \in P$, the set of all vectors $\{w_i - v\}$ forms part of a $\mathbb{Z}$-basis for $\mathbb{Z}^n$ where w_i is the first lattice vector along each edge incident at v. This means $w_i - v$ forms a basis and

$$\det \begin{pmatrix} | & & | \\ w_1 - v & \ldots & w_k - v \\ | & & | \end{pmatrix} = \pm 1$$

where $k \leq n$.

Definition 1.21. A smooth, lattice polytope with lattice-free edges is a *selfatope.*

Note that the simplices in Example 1.7 are all selfatopes.

Definition 1.22. Two selfatopes P and Q in $\mathbb{R}^n$ are *equivalent* if $Q = A \cdot P + \vec{a}$ where A is a $GL_n(\mathbb{Z})$ transformation matrix and $\vec{a}$ is a translation vector in $\mathbb{R}^n$.

Lemma 1.23. *Every selfatope in* $\mathbb{R}^n$ *is equivalent to a selfatope that has* $\vec{0}$ *as one of its vertices.*

Proof. Let v_1 be a vertex of the selfatope $P \in \mathbb{R}^n$. Let $Q = P + (-v_1)$. P and Q are equivalent and Q has its vertex v_1 at $\vec{0}$. $\square$

Definition 1.24. A vertex of a polytope, P is in *standard position* if it lies at $\vec{0}$ and all adjacent edges lie along the axes.

2. The Impossible Selfagons

In the course of our work, we have found certain selfagons (a selfatope $\in \mathbb{R}^2$) do not exist, such as the 5-sided selfagon and the 7-sided selfagon, and while we know very little about why these particular selfagons do not exist and which others are also impossible to construct (for example we have constructed any selfagon with $3n$ sides and also have made a 13-sided selfagon and a 17-sided selfagon), we do have proofs for their nonexistence.

First though, we need some more basic definitions and lemmas before the main results can be stated.

Definition 2.1. Let P be a lattice polygon $\in \mathbb{R}^2$ with vertices $v_i = (x_i, y_i)$ labelled counterclockwise. A vertex v_i is *sharp* $x_i \neq x_{i-1}, x_{i+1}$ and $y_i \neq y_{i-1}, y_{i+1}$ and either

1. $x_i < or > x_{i-1}, x_{i+1}$ and $y_{i-1} < y_i < y_{i+1}$ or $y_{i+1} > y_i > y_{i-1}$
2. $y_i < or > y_{i-1}, y_{i+1}$ and $x_{i-1} < x_i < x_{i+1}$ or $x_{i+1} > x_i > x_{i-1}$

Lemma 2.2. *No selfagon can have a sharp vertex.*

Proof. Assume P is a selfagon and let $v_i \in P$ with adjacent vertices v_{i-1} and v_{i+1} be sharp. If P is a selfagon then it must be smooth at every one of its vertices. Consider the matrix of smoothness at the vertex v_i, and without loss of generality assume that $x_i < x_{i-1}, x_{i+1}$ and $y_{i-1} < y_i < y_{i+1}$. All other cases are analogous to this. The matrix of smoothness is of the form $\begin{pmatrix} x_{i-1} - x_i > 0 & y_{i-1} - y_i < 0 \\ x_{i+1} - x_i > 0 & y_{i+1} - y_i > 0 \end{pmatrix}$. The determinant of this matrix is strictly > 0, so it cannot equal -1 and it has determinant

$+1$ if and only if $x_i = x_{i-1}$ or x_{i+1}, or $y_i = y_{i-1}$ or y_{i+1}. This would contradict the initial assumption of sharpness at v_i. Therefore P cannot be smooth at v_i and therefore cannot be a selfagon. $\square$

Lemma 2.3. *A selfagon can have at most two horizontal (or vertical) edges.*

Proof. Let $P \in \mathbb{Z}^2$ be a selfagon, and let s_1, s_2 be two horizontal edges of P that are not collinear. Note that the third horizontal edge s_3 is also not colinear with either of the other two horizontal edges (if two horizontal edges are colinear then the selfagon is trivially not convex), and order the sides so that the y-value of $s_1 >$ y-value of $s_3 >$ y-value of s_2. Finally, let s_3 connect the vertices v_3, v_4. The edge paths from s_1 to s_3 and from s_2 to s_3 would have to intersect s_3 at different vertices of s_3, so without loss of generality let s_1 be connected to v_3 via an edge path and likewise s_2 to v_4. Then either the line connecting s_1 and v_4 or the line connecting s_2 to $v_3 \not\subseteq P$ and therefore P is not convex. $\square$

Note that the proof that there can be no more than two vertical edges is analogous.

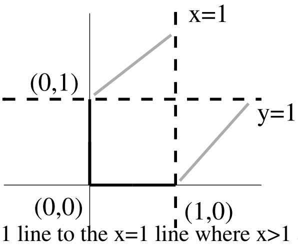
Lemma 2.4. *Any selfagon in the plane is equivalent by a translation and a $GL_2(\mathbb{Z})$ transformation to a selfagon that has a vertex at the origin with adjacent vertices at (0,1) and (1,0).*

Proof. Let P be a selfagon with n vertices. We know that selfagons are translation invariant so P can have its vertex v_1 be anchored at (0,0) with adjacent vertices $v_2 = (a, b), v_n = (c, d)$. Let $v_2 - v_1 = (a, b)$ and $v_n - v_1 = (c, d)$. We know that since P is smooth, (a, b), (c, d) must form a $\mathbb{Z}$-basis of $\mathbb{Z}^2$ and therefore $det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm 1$ which implies that $det \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \pm 1$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \in GL_2(\mathbb{Z})$. Therefore, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} P$ is $GL_2(\mathbb{Z})$ equivalent to P, and the adjacent vertices of (0,0) in P are transformed into $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. $\square$

One of the consequences of Lemma 2.4 is that if no n-sided selfagon exists with a vertex at (0,0) and adjacent vertices (1,0) and (0,1), then there can be no n-sided selfatope in $\mathbb{R}^2$. Now we are ready for one of the main theorems of this section.

Theorem 2.5. *There can be no pentagon $\in \mathbb{R}^2$ that is also a selfagon.*

Proof. Assume that P is a pentagon $\in \mathbb{R}^2$. By the previous lemma, we can say that $v_1 = (0,0)$ with adjacent vertices $v_2 = (0,1), v_5 = (1,0)$. Since $v_2 = (0,1)$ is adjacent to (0,0) its other adjacent vertex v_3 must be on the line $x = 1$ with y-coordinate ≥ 1 in order to preserve smoothness. This is because the matrix of smoothness at v_2 would be $\begin{pmatrix} 0 & -1 \\ x_3 - 0 & y_3 - 1 \end{pmatrix}$ and this can only have determinate ± 1 if $x_3 = \pm 1$. In our construction however, $x_3 \neq -1$ because then P would no longer be convex. The y-coordinate is ≥ 1 because if $v_3 = (1,0)$ then P would be a triangle and not a pentagon at all. Analogously, $v_5 = (1,0)$ is also adjacent to (0,0) and must have its other adjacent vertex, v_4, at some point on the $y = 1$ line with x-coordinate ≥ 1.

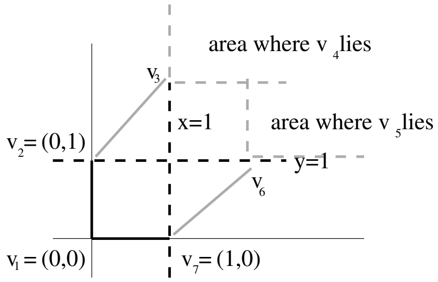


Note that any edge from the y=1 line to the x=1 line where x>1 and y>1 would result in a sharp vertex

If either adjacent vertex to v_2 or v_5 is at (1,1) then P will be a rectangle. Therefore there must be one edge that connects the vertex v_3 on the $x = 1$ line and v_4 on the $y = 1$ line. Since $y_3 \geq 1$ and $x_4 \geq 1$, P is sharp at v_3 and therefore cannot be smooth which directly implies that P cannot be a selfagon. Therefore no pentagon exists which is also a selfagon. ☐

Theorem 2.6. *There can be no* $7 - gon$ *that is also a selfagon.*

Proof. Begin by anchoring the $7 - gon$ Q at $v_1 = (0,0)$ with adjacent vertices $v_2 = (0,1), v_7 = (1,0)$. Let v_3 be the vertex of Q adjacent to v_2 on the $x = 1$ line, $v_3 = (1,a)$ Note that $a > 1$ because if $a = 0$ then Q would be a triangle and if $a = 1$ Q would be a $4 - selfagon$. Analogously, the other adjacent vertex to v_7 on the $y = 1$ line, call it $v_6 = (b,1)$ where $b > 1$. Let v_4 be the other vertex adjacent to v_3. To preserve convexity we can say that $x_4 > 1$. In order to ensure that the vertex v_3 is not sharp, $y_4 \geq a$, and $y_4 = a$ if and only if $a = 2$. This is because if $y_3 = y_4 = a$ the matrix of smoothness at v_3 is of the form $\begin{pmatrix} x_4 - x_3 > 0 & y_4 - y_3 = 0 \\ x_2 - x_3 < 0 & y_2 - y_3 < 0 \end{pmatrix}$ and this has determinant ± 1 if and only if $x_4 - x_3 = 1$ and $y_2 - y_3 = -1$ which implies that $x_4 = 2$ and $y_3 = 2$. Therefore, $v_3 = (1,2)$ and $v_4 = (2,2)$. This implies that Q would already have two horizontal sides and by 0.3 could not have a third. But for Q to be a $7-gon$, $v_4 = (2,2)$ and $v_6 = (b,1)$ must be connected by two edges which is impossible since neither edge can be horizontal, and both edges must vary in the y-direction and therefore cannot cross a span of $y = 1$ which is the distance between y_4 and y_6. Therefore Q could not be a $7-gon$ in this case. By a similar argument $y_5 > 1$ and $x_5 \geq b$, else v_6 would be sharp.



Also v_4 and v_5 must be adjacent, and in order to preserve convexity $y_4 > y_5$ and $x_5 > x_4$. If both not true then the edge connecting v_4 and v_3 would intersect the edge connecting v_4 and v_5 and Q would be a $6-gon$. Otherwise if $x_5 < x_4$ then $tv_6 + (1-t)v_4 \nsubseteq Q$ which implies Q would no longer be convex. This is true because if we let $x_5 < x_4$ and $y_5 < y_4$ then v_6, v_5, v_4 form a triangle in the plane (assume they are not colinear, that would be a trivial case because they wouldn't all be vertices then) with $\overline{v_6v_5}, \overline{v_5v_4} \subseteq$Q but $\overline{v_6v_4} \nsubseteq$Q which would imply that Q is not convex.

So $x_3 - x_4 < 0, y_3 - y_4 < 0, x_5 - x_4 > 0, y_5 - y_4 < 0$ which directly implies that v_4 is sharp and therefore Q cannot be a selfagon. Therefore we cannot have a $7-gon$ selfagon in $\mathbb{R}^2$. ☐

Conjecture 2.7. These selfagons are part of a larger class of impossible selfagons, a class that also includes the 11-gon.

3. The Smooth Planar Fans

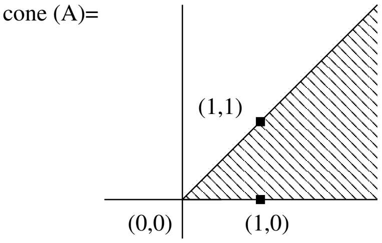
Much of the definitions in this section can be found in Fulton [2] as can further background material on cones and fans. Also, I have adopted his notation for fans and cones.

Definition 3.1. Let $A \subseteq \mathbb{R}^n$. The *cone* spanned by A is:

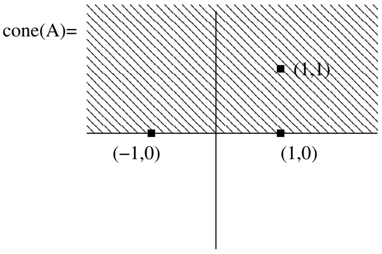
$$cone(A) = \{r_1v_1 + \ldots + r_kv_k | r_i \geq 0,\ v_i \in A\}$$

A *polyhedral cone* is the cone generated by a finite number of vectors, $cone(v_i, \ldots, v_k)$.

Example 3.2. Let A be generated by $\{(1,0),(1,1)\}$



Example 3.3. Let A be generated by $\{(-1,0),(1,0),(1,1)\}$



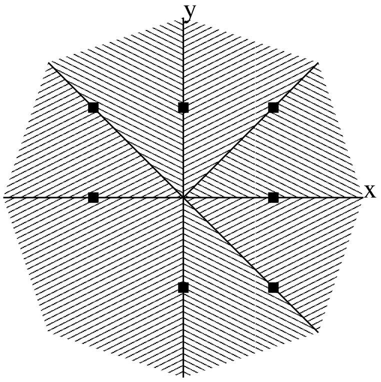
Definition 3.4. A *supporting hyperplane* for a polyhedral cone C is a hyperplane of the form $H = \{\overline{x} \in \mathbb{R}^n | \overline{a} \cdot \overline{x} = 0\}$ with $H \cap C \neq \varnothing$ and $C \subseteq H^+$.

In Example 3.2 the supporting hyperplanes of $cone(A)$ are $y = 0$ and $-y = -x$. In Example 3.3 the supporting hyperplane of $cone(A)$ is $y = 0$.

Definition 3.5. A *polyhedral fan*, $\triangle$, is a finite union of cones satisfying:

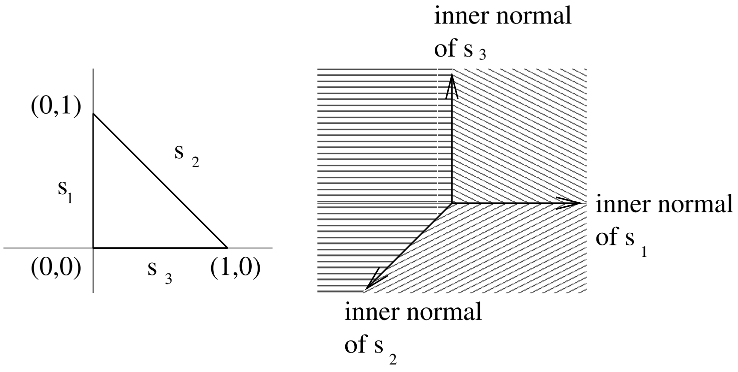
(1) If $\sigma \in \triangle$ and τ is a face of σ, then $\tau \in \triangle$.
(2) If $\sigma,\ \tau \in \triangle$ then $\sigma \cap \tau$ is a face of both σ and τ.

Example 3.6. Let $\triangle$ be generated by (1,0), (1,1), (0,1), (-1,1), (-1,0), (0,-1), (1,-1). Then $\triangle$ is a polyhedral fan and is shown below.



Definition 3.7. Let $P \in \mathbb{R}^n$ be a polytope. The *inner normal fan* $\triangle_P$ of P is the union over all faces $F \subseteq P$ of cones $\sigma_F = \{v \in \mathbb{R}^n | u \cdot v \leq u' \cdot v,\ \forall u \in F, \forall u' \in P\}$.

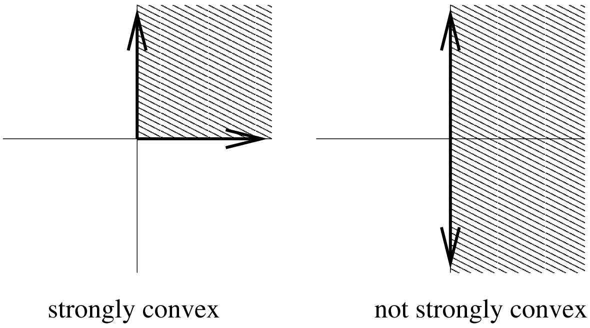
Example 3.8. Let $P = conv\{(0,0),(1,0),(0,1)\}$:



The inner normal fan of P is then shown on the right.

Definition 3.9. A cone $\sigma \in \mathbb{R}^n$ is *strongly convex* if $\vec{0}$ is a face of σ.

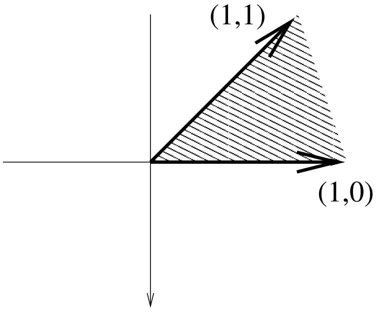
Example 3.10.



Definition 3.11. A *rational polyhedral cone* is a cone whose generators have rational coordinates.

Definition 3.12. Let σ be a rational strongly convex cone in $\mathbb{R}^n$. We say that σ is *smooth* if it can be generated by part of a $\mathbb{Z} - basis$ for $\mathbb{Z}^n$.

Example 3.13. Let $\sigma = cone\{(1,1),(1,0)\}$.



We know σ is smooth because $det \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = -1$, and therefore (1,1) and (1,0) are a $\mathbb{Z} - basis$ for $\mathbb{Z}^2$.

Definition 3.14. A fan $\triangle \in \mathbb{R}^n$ is a *smooth fan* if:

(1) All cones of $\triangle$ are smooth.
(2) Union of all the cones of $\triangle$ is all of $\mathbb{R}^n$.

Note that the fan in Example 3.8 is a smooth fan in $\mathbb{R}^2$.

Lemma 3.15. *Any smooth fan $\triangle \subseteq \mathbb{R}^2$ is the union of a finite number of cones which are generated by a finite number of vectors $\{v_0, \ldots, v_{d-1}, v_d = v_0\}$ (labeled counterclockwise) where v_i, v_{i+1} form a $\mathbb{Z}$-basis for $\mathbb{Z}^n$ and*

(1) *If $A \in GL_2(\mathbb{Z})$ with $det(A) = \pm 1$ then $A \cdot \triangle$ is also a smooth fan.*
(2) *If $A = \begin{pmatrix} v_0 & v_1 \end{pmatrix}^{-1}$ then $A \cdot \triangle$ is smooth and $A \cdot \triangle$ would then have $v_0 = (1,0)$ and $v_1 = (0,1)$.*

Therefore, any smooth fan in $\mathbb{R}^2$ can be transformed into an equivalent smooth fan that has the first quadrant as a cone using a $GL_2(\mathbb{Z})$ transformation matrix.

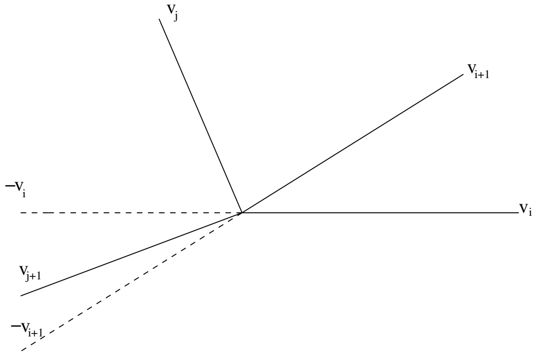
The proof is analogous to the proof of Lemma 2.4

Also, we know the following useful theorems from Fulton [2] hold true for fans in $\mathbb{R}^2$:

Theorem 3.16. *For all $i = 1, \ldots, d$, there exists $a_i \in \mathbb{Z}$ such that $a_i v_i = v_{i-1} + v_{i+1}$.*

Theorem 3.17. *Let v_i and v_{i+1} be any two consecutive rays of $\triangle$ and let v_j be a ray such that $j > i+1$ but v_j lies strictly between v_{i+1} and $-v_i$. Then it is impossible for v_{j+1} to lie strictly between $-v_i$ and*

$-v_{i+1}$.



Theorem 3.18. *For $d \geq 4$, then $v_j = -v_i$ for some i, j.*

Theorem 3.19. $\begin{pmatrix} 0 & -1 \\ 1 & a_1 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & a_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & -1 \\ 1 & a_d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Theorem 3.20. $a_1 + \cdots + a_d = 3d - 12$.

Using these theorems, Sarah Gilles and I will provide an alternate proof that the five-sided selfagon $\in \mathbb{R}^2$ does not exist.

Lemma 3.21. *For any smooth five fan in $\mathbb{R}^2$, we know the following about the five generating rays of the fan, and their corresponding a_i's:*

i. $v_0 = (1,0), v_1 = (0,1)$ *ii.* $v_2 = (-1, a_1)$
iii. $v_3 = (-a_2, a_1a_2 - 1)$ *iv.* $v_4 = (-a_2a_3 + 1, -1)$
v. $a_4 = -a_1a_2 + 1$ *vi.* $a_0 = -a_3a_2 + 1$

Proof. We already know that $v_0 = (1,0)$ and $v_1 = (0,1)$. Let $v_2 = (a,b)$ where $a, b \in \mathbb{Z}$. By Theorem 3.16, there exists an $a_1 \in \mathbb{Z}$ such that $a_1v_1 = v_0 + v_2$. This implies that $a_1(0,1) = (1,0) + (a,b)$ and therefore $(0, a_1) = (a+1, b)$. From this, we can see that $a = -1$ and $b = a_1$. Thus $v_2 = (-1, a_1)$.

Let $v_3 = (c,d)$ with $c, d \in \mathbb{Z}$. Again by Theorem 3.16 there exists $a_2 \in \mathbb{Z}$ such that $a_2v_2 = v_1 + v_3 = (c, d+1)$. Therefore, $c = -a_2$ and $d = a_1a_2 - 1$. Thus $v_3 = (-a_2, a_1a_2 - 1)$.

Let $v_4 = (e,f)$ with $e, f \in \mathbb{Z}$. Because $\triangle$ is smooth, $\det \begin{pmatrix} v_0 & v_4 \end{pmatrix} = \det \begin{pmatrix} 1 & e \\ 0 & f \end{pmatrix} = f = \pm 1$. To preserve convexity of the polytope corresponding to $\triangle$, $f \neq 1$ which implies $f = -1$. By Theorem 3.16 there exists an $a_3 \in \mathbb{Z}$ such that $a_3v_3 = v_4 + v_2$ and thus $(-a_2a_3, a_1a_2a_3 - a_3) = (-1 + e, a_1 - 1)$. Therefore $e = -a_2a_3 + 1$. Consequently, $v_4 = (-a_2a_3 + 1, -1)$.

Also from Theorem 3.16 there exists an $a_4 \in \mathbb{Z}$ such that $a_4 v_4 = v_3 + v_0$. Then $a_4 = -a_1 a_2 + 1$. Furthermore there exists an $a_0 \in \mathbb{Z}$ such that $a_0 v_0 = v_4 + v_1$. Thus $a_0 = -a_3 a_2 + 1$. ☐

Theorem 3.22. *There can be no pentagon that is also a selfagon* $\in \mathbb{R}^2$.

Proof. Let $\triangle$ be a smooth fan generated by five rays $\subseteq \mathbb{R}^2$. From Lemma 3.15, we know we can place v_0 and v_1 at $(1,0)$ and $(0,1)$ respectively.

Because in this case, $d = 5$ we can apply Theorem 3.18, which states there exists i, j such that $v_i = -v_j$ if $d \geq 4$. We will proceed by examining all possible pairings of generating vectors when this theorem is applied.

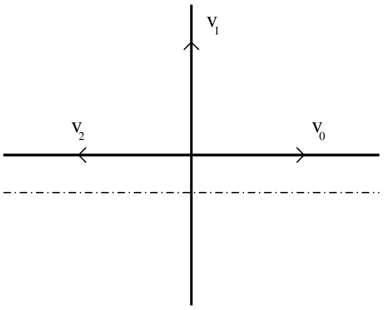
First examine the trivially impossible cases, $v_0 = -v_1$, $v_0 = -v_4$, $v_1 = -v_2$. If $v_2 = -v_3$, this would imply that $(-1, a_1) = (a_2, -a_1 a_2 + 1)$ which then would imply $a_2 = -1$ and therefore $a_1 = a_1 + 1$ which is a contradiction. Thus, this case is also impossible.

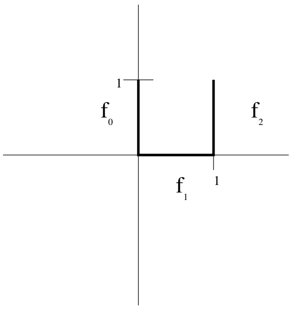
Let $v_0 = -v_2$. This implies that $(1,0) = (1, -a_1)$ and so $a_1 = 0$. This means that $v_2 = (-1,0)$, $v_3 = (-a_2, -1)$ and $a_4 = 1$. From Theorem 3.19 we know

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & a_2 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & a_3 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & -a_3 a_2 + 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} a_2(1-a_3)+1 & a_2(a_2 a_3 - 1)(a_3 - 1) \\ 1 - a_3 & a_2 a_3 (a_3 - 1) + 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus we can see that $a_3 = 1$ which implies $v_4 = (-a_2 + 1, -1)$. It is always true that $v_0 = (1,0)$ and $v_1 = (0,1)$ and from this information, we can generate a general form of the fan for this pentagon.





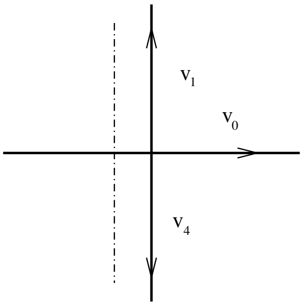
The figure on the left corresponds to the part of the fan generated by this case, where v_3 and v_4 must lie at integer points on the dotted line at $y = -1$. Facets corresponding to the fan's three known rays generate the figure on the right. Note that the sides of the polytope have been scaled down in order to give lattice free edges. Also note that the x-values of f_0 and f_2 vary by one, but the two edges must be connected by two sides of our potential selfagon which is impossible. Hence this fan cannot yield a selfagon.

Also, if we let $v_1 = -v_4$, we see that $(0,1) = (a_2a_3 - 1, 1)$ which implies $a_2a_3 = 1$. Since $a_2, a_3 \in \mathbb{Z}$, we find $a_2 = a_3 = \pm 1$. From Theorem 3.19 we know

$$\begin{pmatrix} 0 & -1 \\ 1 & a_1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & a_2 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & a_2 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & -a_1a_2+1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow$$

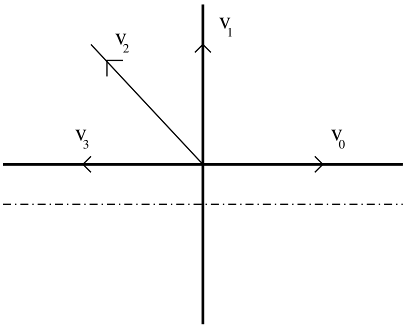
$$\begin{pmatrix} a_1a_2(a_2^2-1) - a_2^2 + a_2 + 1 & a_2^2 - 1 \\ \dots & -(a_1(a_2^2-1) - a_2) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

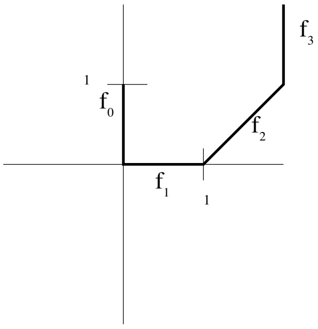
The bottom right position implies that $a_2 = 1$ and so we also know $a_3 = 1$. Therefore, $v_3 = (-1, a_1 - 1)$ while $v_0 = (1,0)$, $v_1 = (0,1)$, $v_2 = (-1, a_1)$ and $v_4 = (1,-1)$.



The fan generated by these five generating vectors is a $GL_2(\mathbb{Z})$ transformation of the fan in case 2 (rotation). Therefore, as when $v_0 = -v_2$, this case cannot yield a selfagon.

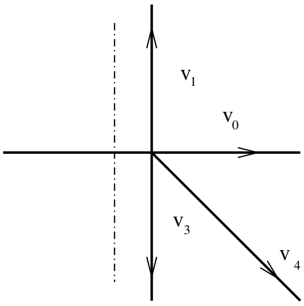
Next, suppose that $v_0 = -v_3$. This would imply $(1,0) = (a_2, -a_1a_2 + 1)$ and thus $a_2 = 1$ and $a_1 = 1$. With this we know that $v_0 = (1,0)$, $v_1 = (0,1)$, $v_2 = (-1,1)$, $v_3 = (-1,0)$ and $v_4 = (-a_3 + 1, -1)$. With these generating vectors, we can create the general form of the fan for this pentagon.





Here the fan on the left corresponds to the part of the inner normal fan generated by these vectors, where v_4 must intersect the $y = -1$ line at a lattice point. The polytope which corresponds to the fan's four known rays is shown at right. Note that the sides of the polytope have been scaled down in order to give lattice free edges. Also note that f_3 and f_0 must be connected with one edge f_4, which would necessarily imply that the vertex at which f_3 and f_4 intersect would be sharp. This implies that the polytope cannot be both smooth and lattice free and therefore this fan cannot generate a five-sided selfagon.

Also if we let $v_1 = -v_3$, we would have $(0, -1) = (-a_2, a_1a_2 - 1)$ and thus $a_2 = 0$. Furthermore, this implies that $v_4 = (1, -1)$ while v_2 remains at $(-1, a_1)$ and we also have $v_0 = (1, 0)$, $v_1 = (0, 1)$, and $v_3 = (0, -1)$.



The fan generated by these five vectors is a $GL_2(\mathbb{Z})$ transformation of the fan in case 3, as it is a reflection across the y-axis composed with a rotation of the original fan. Therefore, as when $v_0 = -v_3$, this case cannot yield a selfagon

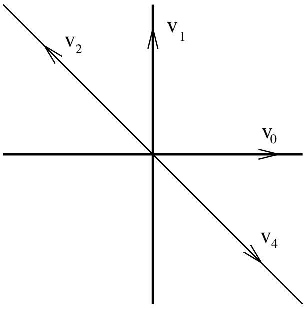
For the next case, suppose that $v_2 = -v_4$. This would imply that $(-1, a_1) = (a_2a_3 - 1, 1)$ and therefore $a_1 = 1$ and $a_2a_3 = 0$. With this we have two cases:

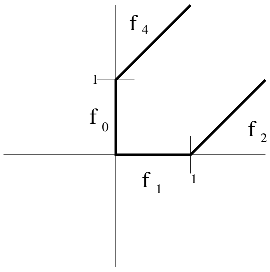
First, let $a_2 = 0$. From statement *v* of Lemma 3.21, we see that $a_4 = 1$ and from statement *vi* of Lemma 3.21, $a_0 = 1$. From Theorem 3.20 we know that $a_1 + a_2 + a_3 + a_4 + a_0 = 3(5) - 12$ and therefore $1 + 0 + a_3 + 1 + 1 = 3$. Ergo, $a_3 = 0$. However,

$$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

By Theorem 3.19, we know this cannot happen and so this case is not possible.

Second, let $a_3 = 0$. From statement *vi* of Lemma 3.21, $a_0 = 1$. Also, since $a_1 = 1$ we can arrive at the conclusion that $a_4 = -a_2 + 1$. With these values for the a_i and our given facts, we know that $v_0 = (1, 0)$, $v_1 = (0, 1)$, $v_2 = (-1, 1)$, $v_3 = (-a_2, a_2 - 1)$, and $v_4 = (1, -1)$. With these generators, we can generate the general fan for this pentagon.



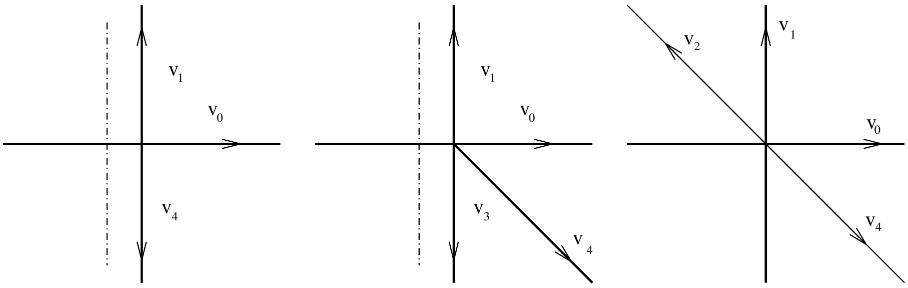


The fan on the left corresponds to the fan generated by the four known vectors, where v_3 lies between v_2 and v_4. The figure on the right corresponds to the polygon that is generated by those four known inner normal vectors. The sides of the polygon have been scaled down to give lattice free edges. Note that the edges f_2 and f_4 must be connected by a single edge f_3. But then the vertex where f_4 and f_3 intersect would be sharp and hence the polygon would not be smooth. Therefore this case cannot produce a selfagon either.

Lastly, suppose $v_3 = -v_4$. From statements *iii* and *iv* of Lemma 3.21 above, $-(-a_2, a_1a_2 - 1) = (-a_2a_3 + 1, -1)$, hence $-a_1a_2 + 1 = -1$. Therefore, $a_1a_2 = 2$. Also, $a_2 = -a_2a_3 + 1$, and since $a_1a_2 = 2$ neither a_1 nor a_2 can be zero, which implies that we can simplify $a_2 = -a_2a_3 + 1$ and divide both sides by a_2. This gives us that $1 = -a_3 + \frac{1}{a_2}$. Also we know that $a_1, a_2, a_3 \in \mathbb{Z}$ which directly implies $a_2 = \pm 1$. Thus $a_1 = \pm 2$. Checking both cases, we find the following values for $a_0, \ldots, a_4$. In the first case, we let $a_1 = 2,\ a_2 = 1,\ a_3 = 0,\ a_4 = -1,\ a_0 = 1$. But then $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and therefore this case is impossible. For the second case, we have $a_1 = -2,\ a_2 = -1,\ a_3 = -2,\ a_4 = -1,\ a_0 = -1$. But then $\begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and therefore this case is also impossible.

□

Lemma 3.23. *Up to $GL_2(\mathbb{Z})$ equivalence, the smooth 5-fans in $\mathbb{R}^2$ must have one of the following three fans as a sub-fan:*



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