The Impossible Selfagons

Selfatopes in the Plane

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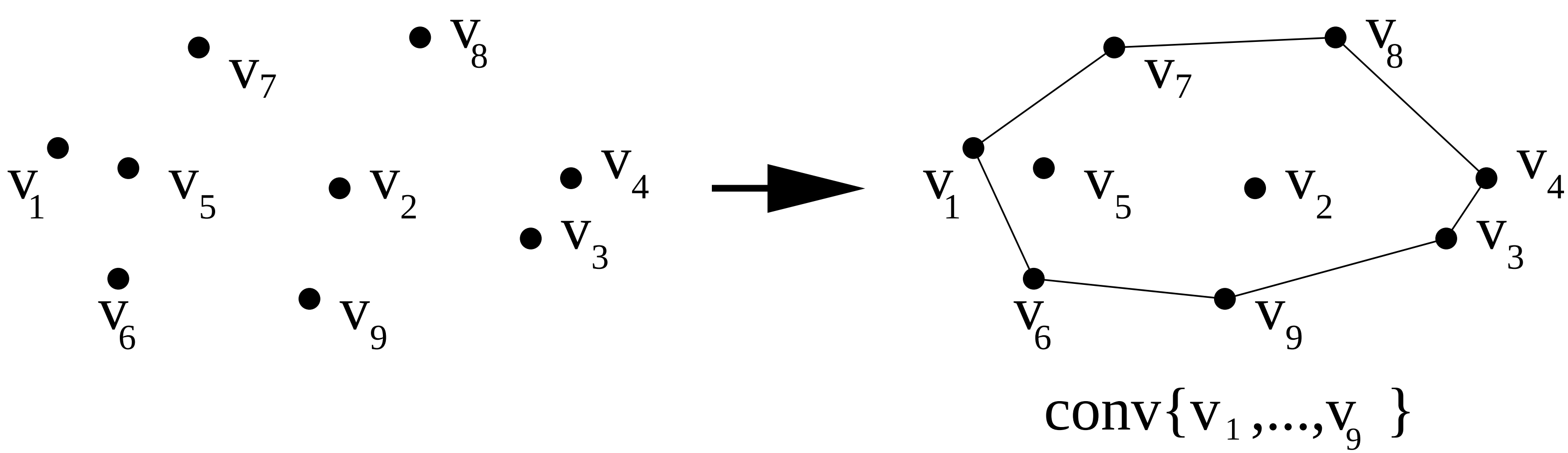
Background and Definitions

Definition 0.1 *The convex hull of a set $A \in \mathbb{R}^n$ is the intersection of all convex sets that contain A.*

Example 0.2



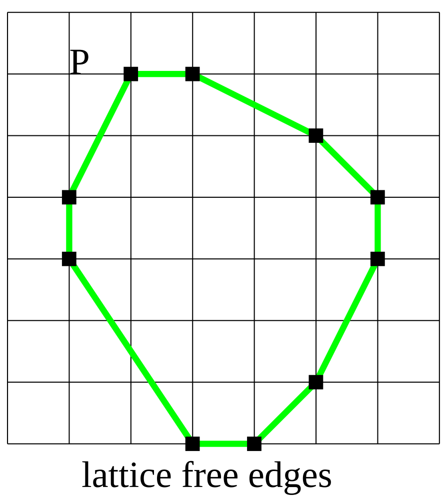
Definition 0.3 *A polytope is the convex hull of a finite set of points in $\mathbb{R}^n$.*

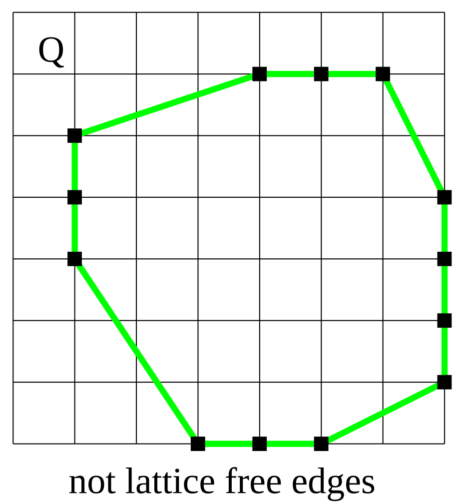
Polytope Properties

Definition 0.4 *A* lattice polytope *is a polytope whose vertices all have integer coordinates.*

Definition 0.5 *Let P be a lattice polytope. Then P has lattice-free edges if the only lattice points on the edges of P are the vertices.*

Example 0.6





More Properties

Definition 0.6.5 Let $P \in \mathbb{R}^2$ be a full dimensionsal lattice polytope with lattice free edges. Let $v_1, \ldots, v_n$ be the vertices of P labelled counterclockwise. Then P is *smooth* if for each vertex $v_i \in P$ the vectors $v_{i-1} - v_i$ and $v_{i+1} - v_i$ forms a $\mathbb{Z}$-basis for $\mathbb{Z}^2$.

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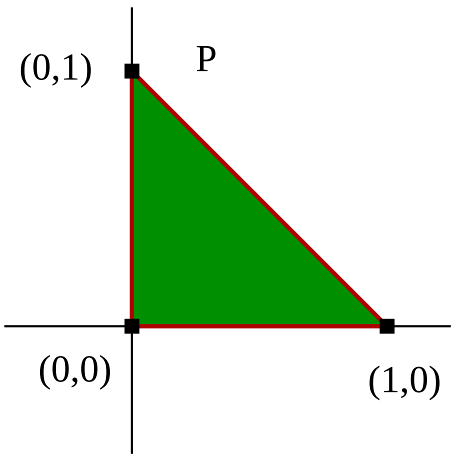
This means that

$$\det \begin{pmatrix} v_{i-1} - v_i \\ v_{i+1} - v_i \end{pmatrix} = \pm 1$$

Smooth

The following is an example of how to calculate if a polytope is smooth



P is smooth. To see this calculate the determinants of the following matrices at each vertex of P:

$$\det \begin{bmatrix} (1,0) - (0,0) \\ (0,1) - (0,0) \end{bmatrix} = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$$

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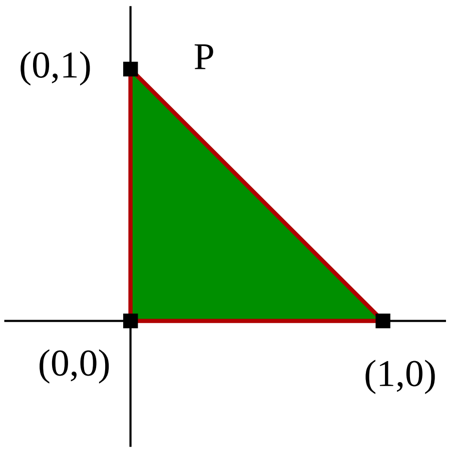

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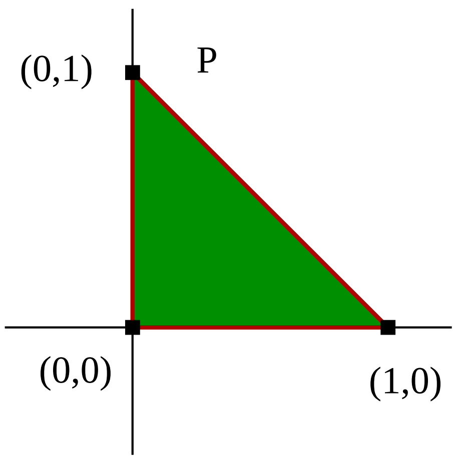

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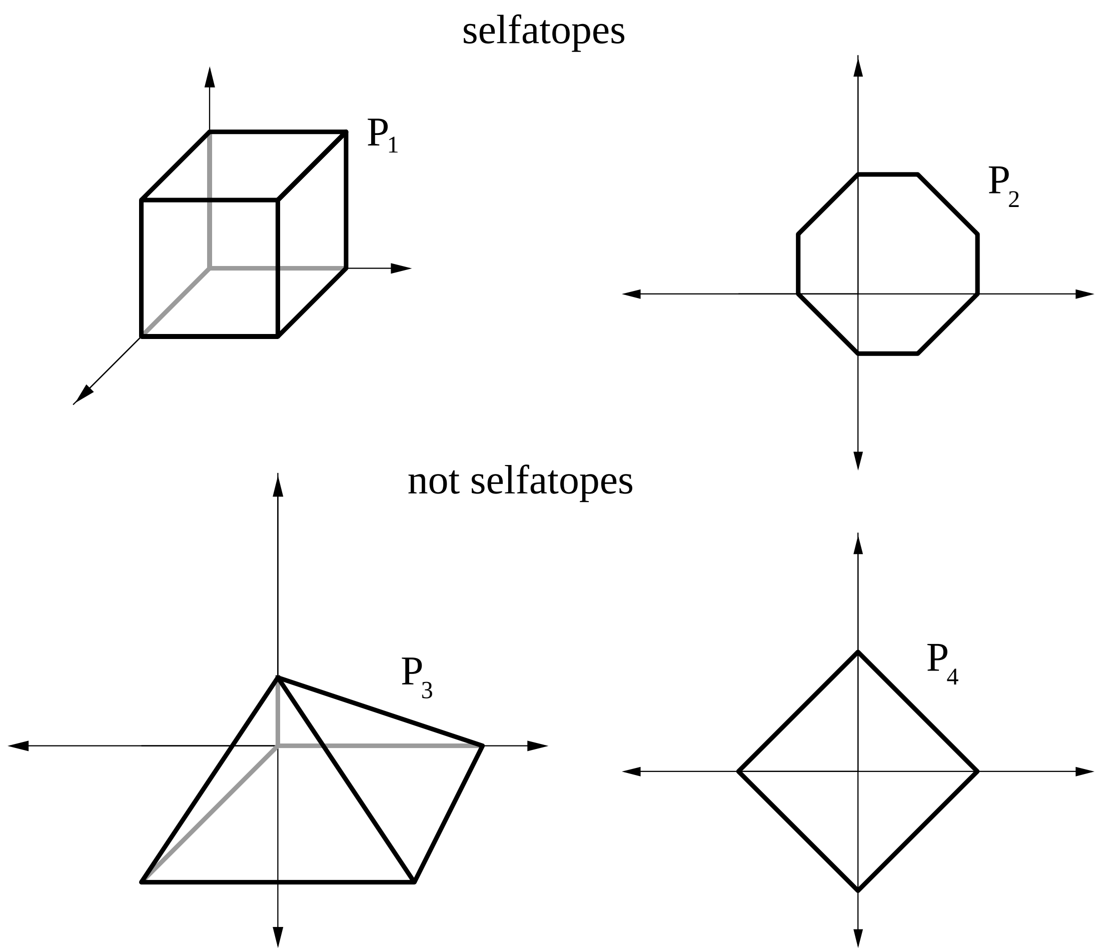
The Selfatope

Definition 0.7 *A smooth, lattice polytope with lattice-free edges is a* selfatope*.*

Example 0.8

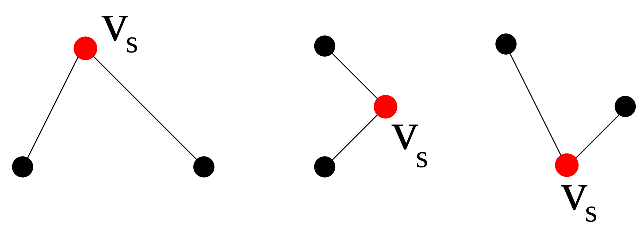


Selfatope Properties

Definition 0.9 *Let P be a lattice polygon $\in \mathbb{R}^2$ with vertices $v_i = (x_i, y_i)$ labelled counterclockwise. A vertex v_i is sharp if $x_i \neq x_{i-1}, x_{i+1}$ and $y_i \neq y_{i-1}, y_{i+1}$ and either*

1. $x_i < or > x_{i-1}, x_{i+1}$ *and* $y_{i-1} < y_i < y_{i+1}$ *or* $y_{i+1} > y_i > y_{i-1}$
2. $y_i < or > y_{i-1}, y_{i+1}$ *and* $x_{i-1} < x_i < x_{i+1}$ *or* $x_{i+1} > x_i > x_{i-1}$

In the figures below, all the vertices labelled v$_s$ are sharp.

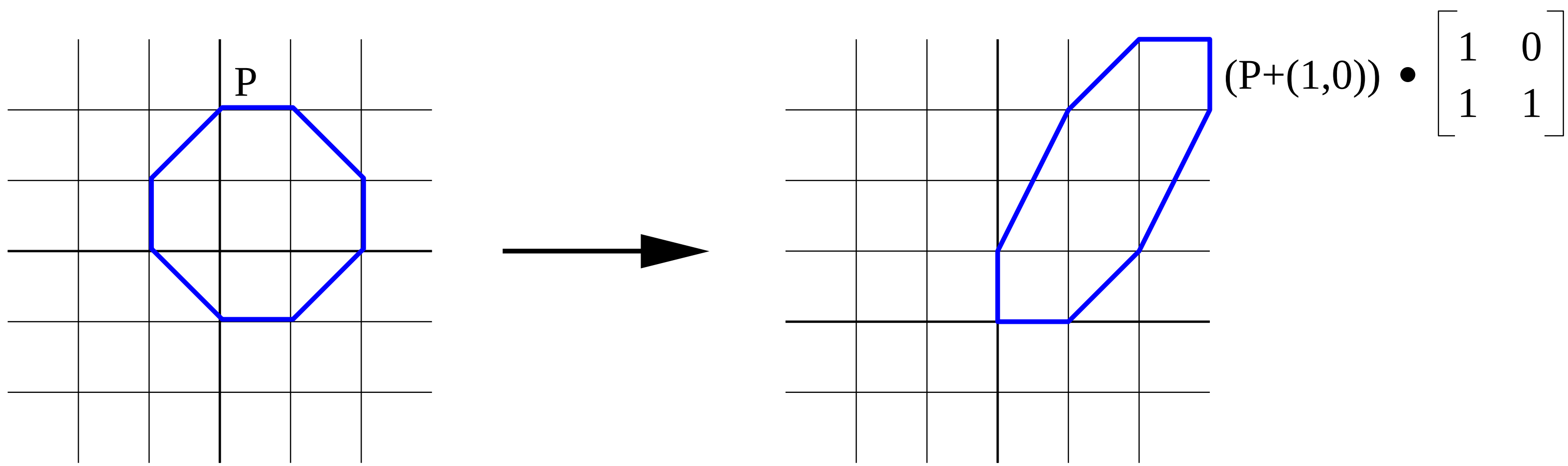


Lemma 0.10 *No selfatope can have a sharp vertex.*

And Even More Properties

Lemma 0.11 *Any selfatope in the plane is equivalent by a translation and a $GL_2(\mathbb{Z})$ transformation to a selfatope that has a vertex at the origin with adjacent vertices at (0,1) and (1,0).*

Example 0.12



Two equivalent selfatopes

The Impossible 5-gon

Theorem 0.13 There can be no 5-sided selfagon in $\mathbb{R}^2$.

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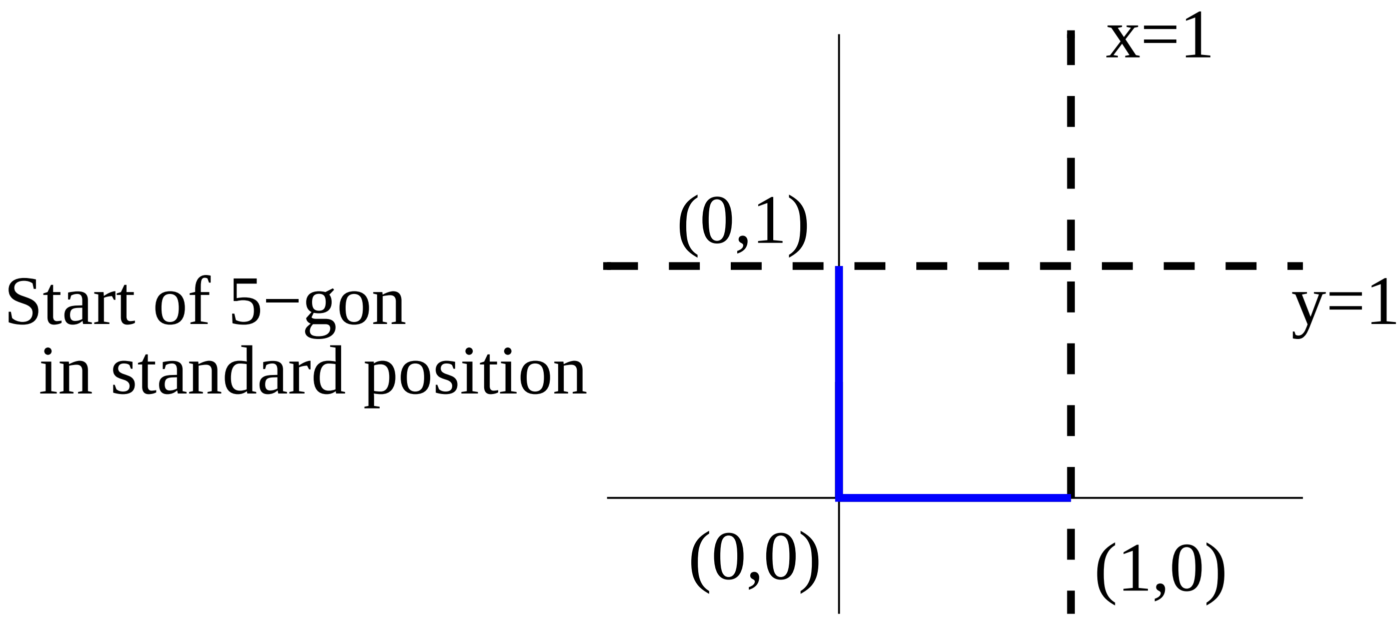

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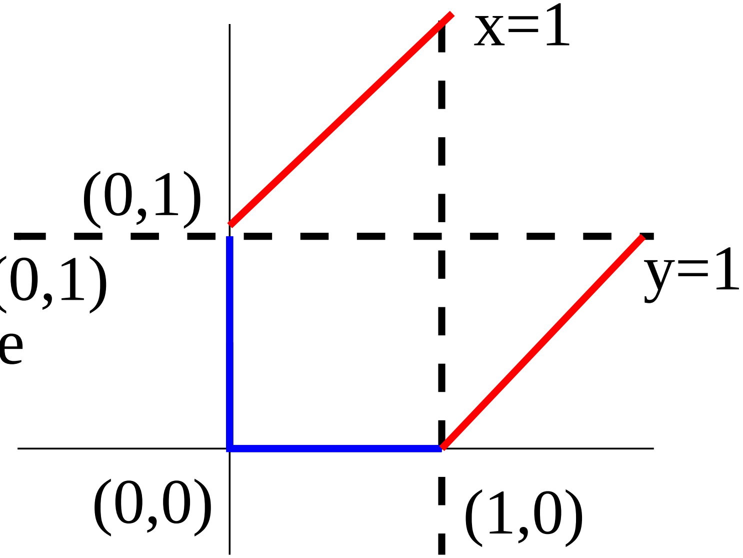

Must connect the vertex at (0,1) to the vertex on the x=1 line

Must connect the vertex at (1,0) to the vertex on the y=1 line

Finding the Impossibles

Theorem 0.14 There can also be no 7-sided selfatope in $\mathbb{R}^2$.

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Theorem 0.14 There can also be no 7-sided selfatope in $\mathbb{R}^2$.

Conjecture 0.15 The 5-sided and the 7-sided selfatopes in $\mathbb{R}^2$ are part of a larger class of Impossible Selfatopes, a class which also includes the 11-gon.

Acknowledgements

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