Secant varieties of toric varieties

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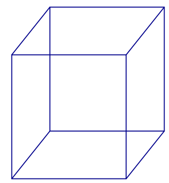
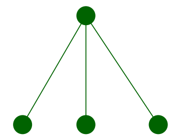
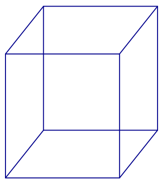
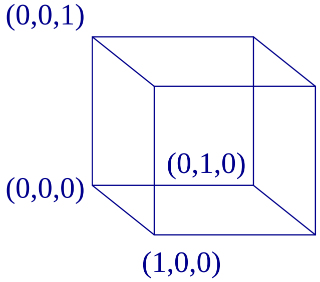
Suitably labeled, the figures  and  represent parametric descriptions of algebraic varieties.

- What are these two varieties and what is the relationship between them?

- What is their relationship to algebraic statistics?

- How does understanding the combinatorics of the cube help us to compute the dimension and degree of the variety associated to the tree?

Claim 1.  $\leftrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.



gives the data of a map $(\mathbb{C}^*)^3 \to \mathbb{P}^7$

$$(s, t, u) \mapsto [1 : s : t : u : st : su : tu : stu].$$

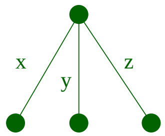
lattice point $(1, 0, 1) \leftrightarrow$ monomial $s^1 t^0 u^1 = su$

In homogeneous coordinates, $s = \frac{x_1}{x_0}, t = \frac{y_1}{y_0}, u = \frac{z_1}{z_0}$

$$[1 : s : t : u : st : su : tu : stu]$$

$$= [x_0 y_0 z_0 : x_1 y_0 z_0 : \cdots : x_1 y_1 z_1].$$

Example 1 (Eriksson, Ranestad, Sturmfels, Sullivant).



root = a past evolutionary state

leaves = currently observable states

π_i = probability of state i at the root

x_{ij} = probability of state changing from i to j along edge x, etc.

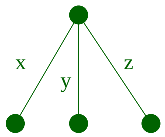
Probability of observing states $i, j, k \in \{0,1\}$

$= \pi_0 x_{0i} y_{0j} z_{0k} + \pi_1 x_{1i} y_{1j} z_{1k}$.

Together these probabilities are the coordinates of a map to $\mathbb{P}^7$:

$$\pi_0 [x_{00}y_{00}z_{00} : x_{01}y_{00}z_{00} : \cdots : x_{01}y_{01}z_{01}]$$

$$+\pi_1 [x_{10}y_{10}z_{10} : x_{11}y_{10}z_{10} : \cdots : x_{11}y_{11}z_{11}].$$



$\leftrightarrow$ the secant variety of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Definition 2. If $X \subset \mathbb{P}^r$, its kth secant variety, $X^{\{k\}}$, is the closure of the union of all $(k-1)$-planes in $\mathbb{P}^r$ that intersect X in at least k points.

What do we expect the dimension of $X^{\{k\}}$ to be?

• If $\dim X = n$, the secant $(k-1)$-planes are spanned by k points of X.

Definition 3. The expected dimension of $X^{\{k\}}$ is

$$\min\{r, (n+1)k - 1\}.$$

Secant varieties are important classically for their role in the theory of projections of varieties.

• If $X \subset \mathbb{P}^r$ can be projected isomorphically to $\mathbb{P}^{r-1}$, then there must exist a point $p \in \mathbb{P}^r$ that does not lie on $X^{\{2\}}$.

• Thus, $\dim X^{\{2\}}$ carries important geometric information about the embeddings of X into projective space.

Theorem 4 (Peters-Simonis, Adlandsvik).

$$\deg X^{\{2\}} \deg \phi$$

$$= (\deg X)^2 - \sum_{k \geq 0} \binom{2n+1}{k} \deg s_k(\Delta, X \times X)$$

Where does this formula come from?

- Embed disjoint copies of X into $\mathbb{P}^{2r+1}$ with coordinates $[x_0 : \cdots : x_r : y_0 : \cdots : y_r]$ and let $J(X, X) \subset \mathbb{P}^{2r+1}$ be the union of all of the lines joining them.

- $\phi : J(X, X) \dashrightarrow X^{\{2\}}$

$$\phi([x_0 : \cdots : x_r : y_0 : \cdots : y_r]) = [x_0 - y_0 : \cdots : x_r - y_r]$$

- The base locus of ϕ consists of points where ϕ is not defined. $s(\Delta, X \times X)$ is the Segre class of the base locus.

- If X is smooth, then this Segre class is just $c(T_X)^{-1} \cap [X]$.

Theorem 5 (Peters-Simonis, Adlandsvik).

$$\deg X^{\{2\}} \deg \phi$$

$$= (\deg X)^2 - \sum_{k \geq 0} \binom{2n+1}{k} \deg(c(T_X)^{-1})_k \cap [X]$$

What is $c(T_X)$?

- If $X = X_P$ is a toric variety $\leftrightarrow P \subseteq \mathbb{R}^n$, then its *Chow ring* can be understood via the polytope P.
- Lattice points of P describe an action of $(\mathbb{C}^*)^n$ on X_P.
- Faces F of $P \leftrightarrow$ torus-invariant subvarieties V_F that generate the Chow ring.
- $c(T_X) \cap [X] = \sum_{F \text{ face of} P} [V_F]$.

Theorem 6. *If $P \subset \mathbb{R}^2$, X_P is smooth and* $\dim X_P^{\{2\}} = 5$, *then*

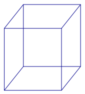
$$\deg X_P^{\{2\}} = \frac{1}{2}(d^2 - 10d + 5B + 2V - 12),$$

where d = area(P), B = # lattice points on the edges, V = # vertices.

Theorem 7. *If $P \subset \mathbb{R}^3$, X_P smooth, and* $\dim X_P^{\{2\}} = 7$, *then*

$$\deg X_P^{\{2\}} = \frac{1}{2}(d^2 - 21d + c_1^3 + 8V + 14E - 84I - 132),$$

where d =volume(P), E = # lattice points on the edges, V = # vertices, and I = # interior lattice points.

Example 2. P =  $d = 1, E = 8, V = 8, I = 0, c_1^3 = 48$

$\deg X_P^{\{2\}} \deg \phi = 36 - 126 + 48 + 64 + 112 - 132 = 2.$

Observations:

- $\deg \phi = 2$ implies that the secant variety of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ fills $\mathbb{P}^7$ exactly once.

- The secant varieties of rational normal scrolls that fill their ambient space also have this property. (See work of Catalano-Johnson.)

- If $A \subset P$, $\dim X_A = \dim X_P$, and $X_A^{\{2\}}$ has the expected dimension, then $X_P^{\{2\}}$ has the expected dimension.

Theorem 8. *If P is a smooth, full-dimensional polytope in $\mathbb{R}^n$, then $X_P^{\{2\}}$ has dimension $2n+1$ and $\deg \phi = 2$ if and only if P is not $AGL(n, \mathbb{Z})$-equivalent to a subset of $2\Delta_n$.*

Key ideas of proof:

- Show that hypotheses imply that P must contain certain "small" polytopes that do not fit inside $2\Delta_n$.

- The most difficult case is when the edges of P are "lattice-free" except for their vertices.

- Lisa Byrne, Sarah Gilles, Vincent Lyzinski, Aaron Wolbach, and Frances Worek studied this special class of polytopes, selfatopes, (smooth, edge lattice-free polytopes) at the Mount Holyoke Summer 2005 REU (DMS-0353700).