AN INTRODUCTION TO TORIC SURFACES

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1. An introduction to affine varieties

To motivate what is to come we revisit a familiar example from high school algebra from a point of view that allows for powerful generalizations.

Example 1.1. In $\mathbb{R}^2$, the parabola is the set of all points satisfying $y = x^2$. We see that for any value of x, there is a unique value of y that satisfies the equation. In fact, we can parameterize the parabola with a map $\phi : \mathbb{R} \to \mathbb{R}^2$ where $\phi(t) = (t, t^2)$.

For each value that the parameter t assumes we get a point in $\mathbb{R}^2$ on the parabola. For example, $\phi(-5) = (-5, 25), \phi(0) = (0, 0), \phi(5) = (5, 25)$ are all points on the parabola. As t varies over all of the real line, ϕ traces out the entire parabola.

In $\mathbb{R}^3$ there is a natural generalization of the parabola called the *twisted cubic*.

Example 1.2. This time, instead of starting off with an implicit equation, we will begin by giving a parameterization of our curve in 3-space. Let $\phi : \mathbb{R} \to \mathbb{R}^3$ be defined by $\phi(t) = (t, t^2, t^3)$.

What does this curve look like? Suppose that you could fly way above the xy-plane and stare straight down in the z-direction. You wouldn't be able to distinguish between points at different heights. In effect, you would just be seeing a curve in the xy-plane given by the parameterization gotten by only looking at the first two coordinates of ϕ. So, you would see the parabola with parameterization (t, t^2).

However, if you were to look at this curve with the xy-plane at eye-level, you would see that when t is positive, the points twist up out of the xy-plane above points on the parabola, and when t is negative, the curve twists down below the xy-plane in an analogous fashion. We see

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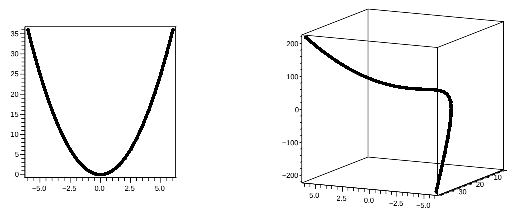
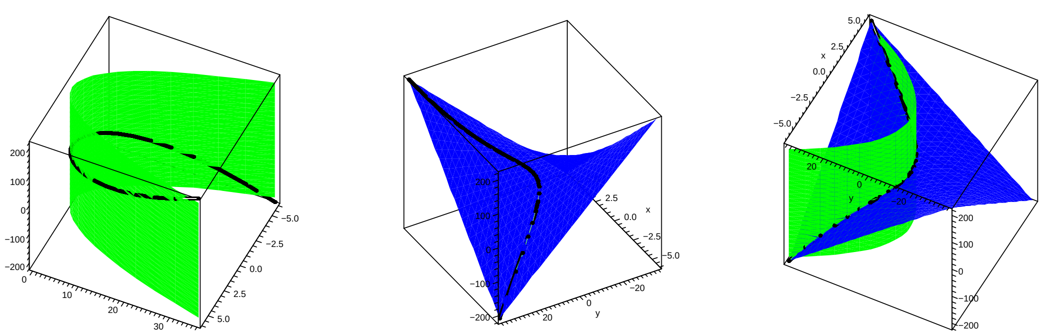
the twisted cubic from two different points of view below.



This parameterized curve can also be defined as the solution set of a system of polynomial equations. We can see that every point satisfies the equation $y = x^2$ because the square of the first coordinate of $\phi(t) = (t, t^2, t^3)$ is equal to its second coordinate.

Note also that the product of the first two coordinates of ϕ is equal to the third. Thus, if we let $f(x, y, z) = xy - z$ we see that $f(\phi(t)) = f(t, t^2, t^3) = t \cdot t^2 - t^3 = 0$. In the picture below we see that the points of the image of ϕ, i.e., the points of $\mathbb{R}^3$ of the form (t, t^2, t^3), are exactly the intersection of the surfaces $y = x^2$ and $xy - z = 0$.



By computing $t^2 - t \cdot t = 0$ and $t \cdot t^2 - t^3 = 0$, we can see that the image of ϕ is contained in the intersection of the solution set of the equations. Morever, suppose that (u, v, w) is a solution of the system of equations. Since (u, v, w) is a solution of the first equation, we see that $v = u^2$, so that $(u, v, w) = (u, u^2, w)$. But the second equation implies that $z = xy$, and this implies that $w = u \cdot u^2 = u^3$. Therefore, any solution of the system of equations given by $y - x^2 = 0$ and $xy - z = 0$ is of the form (u, u^2, u^3) for some $u \in \mathbb{R}$. In other words, all solutions are points in the image of ϕ.

In the examples above, for any real number t, a corresponding point is given explicitly by $\phi(t)$, and as t varies, the points $\phi(t)$ trace out a curve. There is another way of describing these curves. We may realize them implicitly as the solution sets of systems of equations.

The basic object of study in algebraic geometry is a *variety*, or the solution set of a finite system of polynomial equations. We will give a formal definition of a variety that will work in many different settings. To do this, we must introduce some notation.

Let k be a field. We will be interested in the geometry of sets in k^n. Although k^n is a vector space, we will not be performing any vector operations. To emphasize that we are thinking of k^n just as a set of points and not as a vector space, we call it *affine* n-space and write $\mathbb{A}^n_k$ or $\mathbb{A}^n$ if the ground field k is understood. Given indeterminates $x_1, \ldots, x_n$ we write $k[\mathbf{x}] = k[x_1, \ldots, x_n]$ for the ring of polynomials in n variables with coefficients in the field k.

Definition 1.3. Let $f_1(\mathbf{x}), \ldots, f_r(\mathbf{x})$ be polynomials in the ring $k[\mathbf{x}]$. The set

$$V(f_1, \ldots, f_r) = \{\mathbf{p} \in \mathbb{A}^n \mid f_1(\mathbf{p}) = \cdots = f_r(\mathbf{p}) = 0\}$$

is the *affine variety* defined by the equations $f_1(\mathbf{x}) = 0, \ldots, f_r(\mathbf{x}) = 0$.

Example 1.4. Let $f(x, y) = x+y$ and $g(x, y) = x-y$. We can analyze the variety contained in $\mathbb{A}^3_{\mathbb{R}}$ using linear algebra. We know that the solution set of the system of equations

$$\begin{aligned} x+ \qquad & y = 0 \\ x- \qquad & y = 0 \end{aligned}$$

is the same as the solution set of

$$\begin{aligned} x \qquad & \quad = 0 \\ & y = 0. \end{aligned}$$

Thus $V(x + y, x - y) = V(x, y) = \{(0, 0, c) \mid c \in \mathbb{R}\}$.

We can see already with this linear example that different systems of equations may define the same variety. This phenomenon also occurs with varieties defined by higher degree equations.

Example 1.5 (The twisted cubic II)**.** We have seen that the equations $xy - z = 0$ and $y - x^2 = 0$ have common solution set equal to all points of the form (t, t^2, t^3). Looking at the parameterization, we can also see that these points satisfy the equation $y^2 - xz = 0$. In fact, the pair of equations $y^2 - xz = 0$ and $xy - z = 0$ also define the twisted cubic: Suppose that (r, s, t) satisfies both equations. Then $(r, s, t) = (r, s, rs)$ by the second equation. Moreover, by the first, $s^2 - r^2 s = s(s - r^2) = 0$. So, when $s \neq 0$, then $s = r^2$, and we have $(r, s, t) = (r, r^2, r^3)$. When second coordinate is zero, $xy - z = 0$ says that the third coordinate is also zero.

1.1. **The ideal-variety correspondence.** In Example 1.5 we saw that the system of equations defining a variety may not be unique. In fact, it is never unique, even if a variety is defined by a single equation $f(\mathbf{x}) = 0$, we may multiply this equation by any nonzero scalar without changing the solution set.

As the equations defining a variety are not uniquely determined, it makes sense then to consider the set of all equations that the points of a variety satisfy.

Definition 1.6. Let $X \subseteq k^n$. We define $I(X)$ to be the set

$$I(X) = \{f(\mathbf{x}) \in k[\mathbf{x}] \mid f(\mathbf{p}) = 0 \ \forall \ \mathbf{p} \in X\}.$$

Although this set is not finite, we will see that it has the structure of an *ideal*, which can be generated by a finite set of polynomials by the Hilbert Basis Theorem.

Definition 1.7. Let $R = k[\mathbf{x}]$ be a polynomial ring. A nonempty subset I contained in R is an *ideal* if

(1) The sum $f(\mathbf{x}) + g(\mathbf{x}) \in I$ for all $f(\mathbf{x}), g(\mathbf{x}) \in R$.
(2) For each $f(\mathbf{x}) \in I$ and $r(\mathbf{x}) \in R$, $r(\mathbf{x})f(\mathbf{x}) \in I$.

Theorem 1.8. *Let $X \subseteq k^n$. Then $I(X)$ is an ideal.*

Proof. The zero polynomial, $z(\mathbf{x}) = 0$ is clearly in $I(X)$ because it vanishes at every point of k^n. Let $f(\mathbf{x}), g(\mathbf{x}) \in I(X)$. If $\mathbf{p} \in X$, then $f(\mathbf{p}) = g(\mathbf{p}) = 0$. Therefore, when we evaluate $(f + g)(\mathbf{x})$ at $\mathbf{p}$ we have $(f + g)(\mathbf{p}) = f(\mathbf{p}) + g(\mathbf{p}) = 0 + 0 = 0$. Hence, the polynomial $(f + g)(\mathbf{x}) \in I(X)$. Similarly, if $r(\mathbf{x}) \in R$, then $(rf)(\mathbf{x})$ evaluated at $\mathbf{p}$ is $(rf)(\mathbf{p}) = r(\mathbf{x})f(\mathbf{p}) = r(\mathbf{p}) \cdot 0 = 0$. We conclude that $I(X)$ is an ideal. □

We have now seen that we can associate an ideal $I(V)$ to any affine variety V. It is natural to ask if the converse is true: can we associate a variety to every ideal in a polynomial ring? The answer to this question is affirmative, and follows from the Hilbert Basis Theorem.

Theorem 1.9 (Hilbert Basis Theorem)**.** *Let $I \subset k[\mathbf{x}]$ be an ideal in a polynomial ring. Then there exist a finite set $f_1(\mathbf{x}), \ldots, f_m(\mathbf{x})$ such that*

$$I = \{r_1(\mathbf{x})f_1(\mathbf{x}) + \cdots + r_m(\mathbf{x})f_m(\mathbf{x}) \mid r_i(\mathbf{x}) \in k[\mathbf{x}] \ \forall\ i\}.$$

We say that the polynomials $f_1(\mathbf{x}), \ldots, f_m(\mathbf{x})$ are a finite set of generators *for I. We write $I = \langle f_1(\mathbf{x}), \ldots, f_m(\mathbf{x})$ rangle.*

Example 1.10 (Parabola)**.**

Example 1.11 (Twisted cubic III)**.**

Definition 1.12. Suppose that I is an ideal in $k[\mathbf{x}]$. The *variety* associated to I is the set

$$V(I) = \{\mathbf{p} \in k^n \mid f(\mathbf{p}) = 0 \ \forall f(\mathbf{x}) \in I\}.$$

Theorem 1.13. *If $I \subseteq k[\mathbf{x}]$ is an ideal, then $V(I)$ is an affine variety.*

Proof. By the Hilbert Basis Theorem, $I = \langle f_1(\mathbf{x}), \ldots, f_m(\mathbf{x})$ *rangle.* □

1.2. Singularities.

1.3. Dimension.

2. Projective space

2.1. Homogeneous coordinates.

2.2. An open affine covering of projective space.

2.3. Projective varieties and homogeneous ideals.

2.4. Limit points.

2.5. Degree.

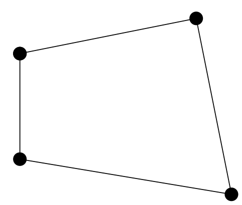
3. Polytopes

We are familiar with two and three-dimensional polytopes. For example, triangles, rectangles, and pentagons are all two-dimensional polytopes, or *polygons*. In dimension three we have the tetrahedron, the cube, and many other three-dimensional polytopes.

The geometry and combinatorics of polytopes is an interesting subject in its own right. For us, the point of interest is that polytopes carry a surprising amount of information about a certain class of varieties called projective toric varieties. We have seen that the algebraic definitions of some important properties of varieties are very difficult to work with in practice. However, there is a dictionary relating properties such as smoothness, dimension, and degree for toric varieties to properties of polytopes that are easier to compute.

3.1. **Polytopes as convex hulls.** Although we will mainly be interested in 2-dimensional polytopes, i.e., polygons, for which most of us have a lot of intuition, we will give careful definitions for arbitrary dimensions as we will need them when we begin to decompose our polygons.

Perhaps the most basic issue to address is: How do we describe a polygon? Note that we need only know the vertices. In the plane we can construct a polygon from vertices by drawing edges between vertices in such a way that we enclose a region P that is *convex*, i.e., if $p, q \in P$ then so is the line segment joining p and q.

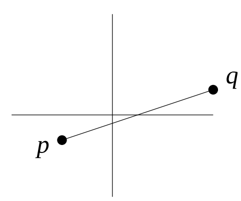


To think about how to make this process rigorous in general, let us recall how to parameterize the line joining two points.

Example 3.1. Let $p = (-2, -1)$ and $q = (3, 1)$ in the plane. Our parameter t will vary in the interval $0 \leq t \leq 1$. The formula

$$\begin{aligned} \ell(t) &= (1-t)p + tq \\ &= (1-t)(-2,-1) + t(3,1) \\ &= (-2+2t, -1+t) + (3t, t)) \\ &= (-2+5t, -1+2t) \end{aligned}$$

parameterizes a line segment in $\mathbb{R}^2$ joining p and q as we see that $\ell(0) = (-2, -1) = p$ and that $\ell(1) = (3, 1) = q$.



We want a generalization of this idea to an arbitrary finite set of points. It may help to think of the construction in Example 3.1 in the following way. Given two points p and q, $\ell(t)$ is given by linear combinations $sp + tq$ in which the values of r and q are constrained in two ways. First, s is really a function of t where $s = (1-t)$. This relation between the coefficients explains why $\ell(t)$ is one-dimensional instead of two-dimensional. We have also bounded the value of t between 0 and 1. This explains why $\ell(t)$ just describes a line segment instead of a whole line.

The line segment $\ell(t)$ is the smallest convex region containing the points p and q. The following definition tells us how to construct the smallest convex region containing an arbitrary finite set of points in $\mathbb{R}^n$.

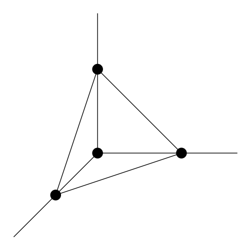
Definition 3.2. Let $\mathcal{V} = \{p_1, \ldots, p_n\} \subset \mathbb{R}^m$. The *convex hull* of $\mathcal{V}$ is the set

$$\operatorname{conv} \mathcal{V} = \{r_1 p_1 + \cdots + r_n p_n \mid r_1 + \cdots + r_n = 1, 0 \leq r_i \leq 1\}.$$

The set $\operatorname{conv} \mathcal{V}$ is a $\mathcal{V}-$*polytope*.

Example 3.3 (The cube)**.** The square is a 2-dimensional cube and is equal to $\operatorname{conv}\{(0,0),(1,0),(0,1),(1,1)\}$. The n-dimensional cube may be realized as the convex hull of the 2^n 0-1 vectors in $\mathbb{R}^n$.

Example 3.4 (The simplex)**.** The 2-dimensional standard simplex is $\operatorname{conv}\{(0,0),(1,0),(0,1)\}$. The n-dimensional simplex is the convex hull of the standard basis vectors together with the origin.



Note that this definition ensures that the resulting set is indeed convex.

Proposition 3.5. *Let P be the convex hull of points $p_1, \ldots, p_n$. The line segment joining any pair of points in P is contained in P.*

Proof. Suppose that p and q are in P. This implies that $p = \sum r_i p_i$ and $q = \sum s_i p_i$ where $0 \leq r_i, s_i, \leq 1$ and $\sum r_i = \sum s_i = 1$. A point on the line joining p and q has the form

$$\begin{aligned} p + (1-t)q &= tr_1 p_1 + \cdots tr_n p_n + (1-t)s_1 p_1 + \cdots, +(1-t)s_n p_n \\ &= (tr_1 + (1-t)s_1)p_1 + \cdots + (tr_n + (1-t)s_n)p_n. \end{aligned}$$

As $0 \leq t \leq 1$, we see that $0 \leq tr_i, (1-t)s_i \leq 1$. Moreover,

$$\begin{aligned}
&(tr_1 + (1-t)s_1) + \cdots + (tr_n + (1-t)s_n) \\
&= t(r_1 + \cdots + r_n) + (1-t)(s_1 + \cdots + s_n) \\
&= t \cdot 1 + (1-t) \cdot 1 \\
&= 1.
\end{aligned}$$

□

Theorem 3.6. *Let* $\mathcal{V} = \{p_1, \ldots, p_n\} \subset \mathbb{R}^m$. *The* convex hull *of* $\mathcal{V}$ *is the smallest convex set containing* $\mathcal{V}$. *In other words, if* $C \subset \mathbb{R}^m$ *is a convex set containing* $\mathcal{V}$, *then* $\operatorname{conv} \mathcal{V} \subseteq C$.

Proof. See the discussion on pages 3 and 4 of [**?**]. □

Up until now we have relied upon an intuitive notion of dimension, but we will need a rigorous definition. Since our definition of the convex hull of a finite set of points is based upon taking linear combinations it makes sense that we may define dimension in terms of linear algebra.

Definition 3.7. Let $P \subseteq \mathbb{R}^m$ be the convex hull of points $p_1, \ldots, p_n$. The *affine span* of P is

$$\operatorname{aff} P = \{r_1 p_1 + \cdots r_n p_n \mid r_1 + \cdots + r_n = 1, r_i \in \mathbb{R}\}.$$

To motivate the definition, let us consider the example of a line joining two points again.

Example 3.8. Again, let $p = (-2, -1)$ and $q = (3, 1)$ in the plane. If we allow our parameter t to range over all of $\mathbb{R}$ then $\ell(t) = (1-t)p + tq = (-2 + 5t, -1 + 2t)$ parameterizes the line joining $(-2, -1)$ and $(3, 1)$ defined by $y = \frac{2}{5}x - \frac{1}{5}$. We can translate this line to the origin by subtracting the point $(-2, -1)$ from each point to get $(5t, 2t) = t(5, 2)$, which is the span of a single nonzero vector and hence is one-dimensional

The affine span of a set of points is just a translation of a linear subspace so that it does not necessarily contain the origin. If we think of the affine span in this way, then it makes sense to define dimension by translating P to the origin and letting the dimension of the affine span be the dimension of the linear subspace through the origin.

Observe that if, say, the point p_n is at the origin, then

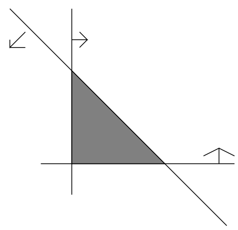
$$\{r_1 p_1 + \cdots r_n p_n \mid r_1 + \cdots + r_n = 1, r_i \in \mathbb{R}\}.$$

is just the span of the first $n - 1$ points as the value of r_n is irrelevant, allowing $r_1, \ldots, r_{n-1}$ to vary freely.

Definition 3.9. Let $P = \mathrm{conv}\{p_1, \ldots, p_n\} \subset \mathbb{R}^m$. Then the *dimension* of P is the dimension of the linear subspace $\mathrm{aff}(P - p_n)$.

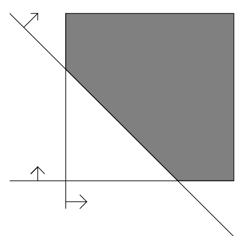
3.2. **Polytopes as intersections of halfspaces.** We have a good intuitive idea for the vertices and edges of a polygon in the plane. However, we will need a rigorous definition of these ideas that generalizes to higher dimensions. For this, an alternative way of describing polytopes via intersections of halfspaces will be useful.

Example 3.10 (The simplex revisited)**.** Consider the following inequalities: $x \geq 0, y \geq 0, x + y \leq 1$.



Definition 3.11. Suppose we are given a set of vectors $\mathbf{n}_1, \ldots, \mathbf{n}_k \subset \mathbb{R}^m$ and real numbers $a_i \in \mathbb{R}$. The vectors in $\mathbb{R}^m$ that satisfy the inequalities $\mathbf{x} \cdot \mathbf{n}_i \geq a_i$ for all i form an $\mathcal{H}$- *polyhedron.* A bounded $\mathcal{H}$-polyhedron is an $\mathcal{H}$-*polytope.*

Example 3.12 (Unbounded polyhedra)**.** Consider the linear inequalities $x \geq 0$, $y \geq 0$, $x + y \geq 1$. The resulting $\mathcal{H}$-polyhedron depicted below is unbounded.



Theorem 3.13. *Every $\mathcal{H}$-polytope is a $\mathcal{V}$-polytope and every $\mathcal{V}-$polytope is an $\mathcal{H}$-polytope. In other words, a polytope can be constructed from an intersection of halfspaces or as the convex hull of a finite set of points.*

Proof. See section 1.1 of [**?**]. □

Example 3.14 (Equations for the standard simplex in $\mathbb{R}^n$)**.** Equations for the standard simplex are given by $x_i \geq 0$ for each $i = 1, \ldots, n$ and $-x_1 - \cdots - x_n \geq 1$.

Example 3.15 (Equations for the n-dimensional cube)**.** Equations are given by $x_i \geq 0$ and $-x_i \geq 1$ for each $i = 1, \ldots, n$.

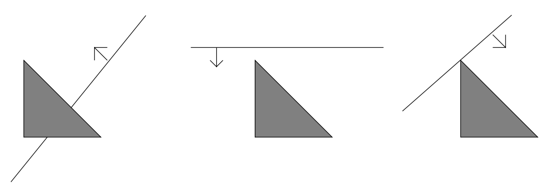
Theorem 3.16.

3.3. **The faces of a polytope.** Now that we are able to describe polytopes as the intersection of halfspaces, we can give a rigorous definition of a *face*. It is instructive to think about the 2-dimensional case first.

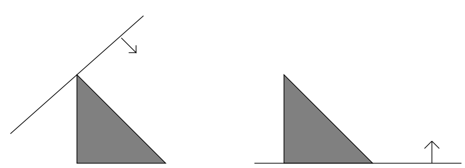
Example 3.17 (Faces of the simplex in $\mathbb{R}^2$)**.** . Consider the standard simplex given by $x \geq 0, y \geq 0, x + y \leq 1$. We want to define faces so that we have three edges, or 1-dimensional faces, and three vertices, or 0-dimensional faces.

What property distinguishes points on faces from points in the interior of the simplex? Suppose we consider an arbitrary halfspace $ax + by \geq c$. The relationship between such a halfspace and our simplex falls into three different cases: the simplex does not lie in the halfspace, the simplex lies entirely in the interior of the halfspace, the simplex intersects the boundary of the halfspace and is contained in the halfspace.

These situations are depicted below:



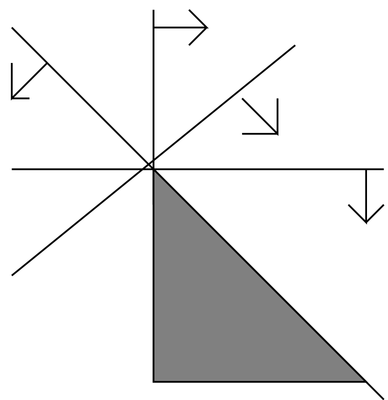
It is the third situation that interests us here. If the simplex lies in a halfspace and meets its boundary, then the intersection of the two is either an edge or a vertex.



Definition 3.18. Let $P \subseteq \mathbb{R}^n$ be a polytope. We say that $\mathbf{b} \cdot \mathbf{x} = c$ is a *supporting hyperplane* of P if P is contained in the halfspace $\mathbf{b} \cdot \mathbf{x} \geq c$ and there exists a nonempty subset of points $p \in P$ such that equation $\mathbf{b} \cdot \mathbf{p} = c$. In other words, P intersects the boundary of the halfspace. If H is a supporting hyperplane of P, then $H \cap P$ is a *face* of P.

Example 3.19 (Supporting hyperplanes for the simplex)**.** The simplex Δ in the plane has three edges and three vertices. It is easy to see that for each edge e there is exactly one supporting hyperplane H_e such that $H_e \cap \Delta = e$.

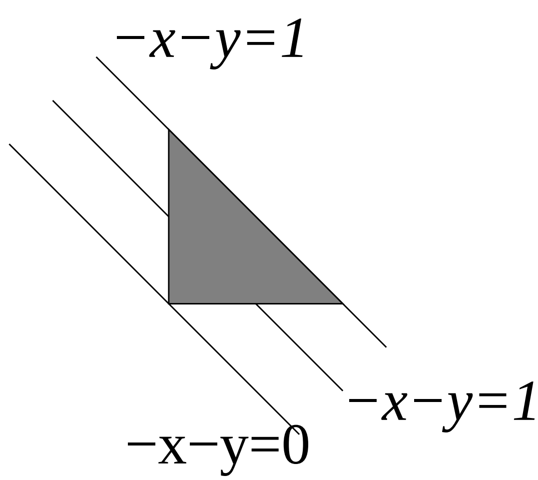
It is more interesting to try to describe the supporting hyperplanes of a vertex. In the diagram below we depict several supporting hyperplanes which intersect the simplex in the same vertex.



Notice that each edge of a polygon is associated to a single supporting hyperplane, but that a vertex is associated to infinitely many supporting hyperplanes. We next seek a way of organizing information about how supporting hyperplanes are related to one another.

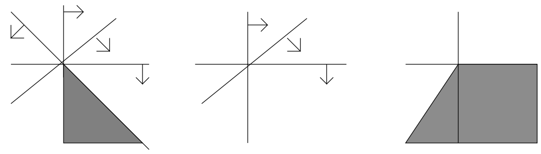
To do this, we will think about multivariable calculus.

Example 3.20. Consider the function $f(x, y) = -x - y$. Its level sets are given by equations of the form $x + y = c$, where c is a constant. The value of c for which $f(x, y)$ achieves its maximum on the simplex is exactly when $c = 1$ and $-x - y = c$ is a supporting hyperplane.



If we want to talk about this family of hyperplanes, or lines, all we need to do is give the normal vector $(-1, -1)$.

Let us determine the set of vectors normal to the supporting hyperplane of the origin and pointing into the simplex so that the supporting hyperplane maximizes a linear function on the simplex.



We see that if we translate the inward pointing normal vectors of the supporting hyperplanes of the vertex to the origin we get a region that looks like a cone that lies between the rays spanned by $(1, 0)$ and $(-1, -1)$.

3.4. **The inner normal fan of a polytope.** To discuss supporting hyerplanes of vertices, and more generally, of faces of codimension greater than 1, we need a rigorous definition of a cone.

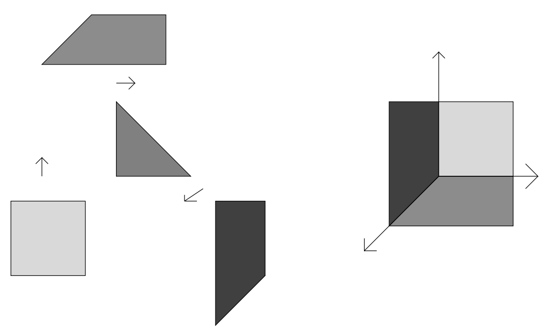
Definition 3.21. The *cone* generated by $\mathcal{A} = \{\mathbf{p}_1, \ldots, \mathbf{p}_n\} \subset \mathbb{R}^m$ is the set

$$\operatorname{cone} \mathcal{A} = \{r_1\mathbf{p}_1 + \cdots r_n\mathbf{p}_n \mid 0 \leq r_i\}.$$

A cone may also be described as the intersection of halfspaces. Thus we can define supporting hyperplanes and faces of a cone in the same way that we defined them for polytopes.

If P is a polytope and F is a face of P, then the inner normal vectors of supporting hyplanes of P intersecting F form a cone. These cones fit together in a nice way.

Example 3.22. Below we depict the simplex in the plane together with the cone of inner normals of supporting hyerplanes at each face.



Notice that the cones fit together in such a way that the intersection of any pair of cones is another cone that is a face of each of them.

This brings us to the definition of a *fan*.

Definition 3.23. A *polyhedral fan* Δ is a collection of cones with the property that if $\sigma, \tau \in \Delta$ then $\sigma \cap \tau$ is a face of each and is also a cone in Δ.

Theorem 3.24. *The cones of inner normals of supporting hyperplanes of a polytope P fit together in a fan called* the inner normal fan *of P.*

3.5. **Exercises.**

(1) Draw the convex hull of each set below.
  (a) $\{(0,0),(1,0),(0,1)\}$
  (b) $\{(0,0),(2,0),(0,2)\}$
  (c) $\{(0,0),(1,0),(1,2)\}$
  (d) $\{(m,0) \mid m \in \mathbb{Z}_{>0}\} \cup \{(0,m) \mid m \in \mathbb{Z}_{>0}\}$
(2) Draw $\text{conv}\{(0,0),(1,0),(2,0),(0,1),(1,1)\}$. Find a subset of the given points with the same convex hull.
(3) Draw $\text{conv}\{(1,1,0),(1,-1,0),(-1,1,0),(-1,-1,0),(0,0,1),(0,0,-1)\}$.
(4) Let $p = (1,0,0)$ and $q = (0,1,0)$.
  (a) Draw the linear span of p and q.
  (b) Draw the affine span of p and q.
  (c) Can you explain the difference that you see?
(5) Consider the triangle $P = \text{conv}(((0,0),(2,0),(0,1),(1,1))$.
  (a) Draw contours of the function $f(x,y) = x$. At which points of P does f attain its maximum? its minimum?
  (b) Repeat this for the function $g(x,y) = y$.
  (c) Now consider an arbitrary linear polynomial $L(x,y) = ax+by$. Can you describe the points of P at which L attains its minimum in terms of a and b?
(6) Consider the inequalities $x \geq 0, y \geq 0, -y \geq a, and x - y \geq -b$.
  (a) Draw the subset of $\mathbb{R}^2$ satisfying the inequalities if $a = 1$ and $b = 2$.
  (b) Draw the subset of $\mathbb{R}^2$ satisfying the inequalities if $a = 3$ and $b = 2$.
  (c) Draw the subset of $\mathbb{R}^2$ satisfying the inequalities if $a = 1$ and $b = 3$.
  (d) Do the three polyhedra above have the same inner normal fans?

4. Affine toric varieties: from lattice points to monomial mappings

In this chapter we introduce toric varieties embedded in affine space. We begin by giving embeddings and then show how to compute the ideal of an affine toric variety from its parameterization. Throughout sections 4.2 and 4.3, we follow Sturmfels's book [3].

Notice that in the two examples above, the parameterizations had a particularly nice form; each coordinate was just a power of t. The implicit equations defining these two varieties also had a special form; a monomial (product of indeterminates) minus another monomial.

In this section we will define *affine toric varieties* which may be viewed as higher dimensional analogues of these examples as they are parameterized by maps with monomial coordinates and may be cut out by binomial implicit equations.

4.1. **Notation.** We introduce the notation that we will use to describe our embeddings.

Definition 4.1. We call $\mathbb{Z}^d \subset \mathbb{R}^d$ the d-dimensional *lattice* and call elements of $\mathbb{Z}^d$ *lattice points.*

Example 4.2. For example, $(0,0),(1,0),(2,1)$ are lattice points in $\mathbb{R}^2$.

Definition 4.3. Suppose that we have indeterminates $x_1,\ldots,x_d$. If we have a lattice point $\mathbf{a} = (a_1,\ldots a_d)$ then we can associate to it a *Laurent monomial* in the x_i's: $x_1^{a_1}\cdots x_n^{a_d}$. (We call this a *Laurent* monomial because we allow negative exponents.) We often denote this Laurent monomial by $\mathbf{x}^{\mathbf{a}}$ using vector notation instead of writing out each indeterminate and each exponent.

Example 4.4. Let $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1,x_2,x_3]$. Then $\mathbf{x}^{\begin{pmatrix}4 & 1 & 2\end{pmatrix}} = x_1^4x_2^1x_3^2$.

4.2. **The variety $X_{\mathcal{A}}$..** Suppose we have a finite ordered set containing n elements, $\mathcal{A} = \{\mathbf{a}_1,\ldots,\mathbf{a}_n\} \subset \mathbb{Z}^d$. We can use $\mathcal{A}$ to define a map to $\mathbb{C}^n$ whose coordinates are the monomials $\mathbf{t}^{\mathbf{a}}$. Since we allow monomials with negative exponents, none of the elements in the domain can have a zero coordinate.

Definition 4.5. Let $\mathbb{C}^* = \mathbb{C}\backslash\{\mathbf{0}\}$. Given $\mathcal{A} = \{\mathbf{a}_1,\ldots,\mathbf{a}_n\} \subset \mathbb{Z}^d$, we define $\phi_{\mathcal{A}} : (\mathbb{C}^*)^d \to \mathbb{C}^n$ by $\mathbf{t} = (t_1,\ldots,t_d) \mapsto (\mathbf{t}^{\mathbf{a}_1},\ldots,\mathbf{t}^{\mathbf{a}_n})$.

Example 4.6. Suppose that $\mathcal{A} = \{1,2\} \subset \mathbb{Z}$. Then $\phi_{\mathcal{A}}(t) = (t,t^2)$. Restrict the domain to real numbers and think about the image of this map in $\mathbb{R}^2$. It is a parameterization of the parabola in the plane minus the point at the origin as in Example 1.1.

Example 4.7. Suppose that $\mathcal{A} = \{1,2,3\} \subset \mathbb{Z}$. Then $\phi_{\mathcal{A}}(t) = (t,t^2,t^3)$. Again, let us restrict the domain to real numbers and try to visualize what we get in $\mathbb{R}^3$. We see that we get the twisted cubic of Example 1.2

Definition 4.8. The *affine toric variety* associated to $\mathcal{A}$, denoted $X_{\mathcal{A}}$, is the Zariski closure of the image of $\phi_{\mathcal{A}}$. The image of $\phi_{\mathcal{A}}$ is a dense open subset of $X_{\mathcal{A}}$.

In our two previous examples, the variety $X_{\mathcal{A}}$ is gotten by adding in the origin. Here is a more interesting example in which we add more than one point when we take the closure of the image of $\phi_{\mathcal{A}}$.

Example 4.9. Let $\mathcal{A} = \{(1,0),(0,1),(1,1)\}$. Then $\phi_{\mathcal{A}} : (\mathbb{C}^*)^2 \to \mathbb{C}^3$ is given by $\phi_{\mathcal{A}}(t_1,t_2) = (t_1,t_2,t_1t_2)$. Consider points of the form $(0,y,0)$, where $y \in \mathbb{C}^*$. If we let $t_2 = y$ and consider $\lim_{t_1\to 0}(t_1,t_2,t_1t_2)$ in the complex numbers, we get $(0,y,0)$. As the Zariski closure of a set is closed in the usual topology, these limit points must be in $X_{\mathcal{A}}$. Similarly, the line $\{(x,0,0) \mid x \in \mathbb{C}\}$ is contained in $X_{\mathcal{A}}$.

We can visualize the real points in the image of $\phi_{\mathcal{A}}$ in $\mathbb{R}^3$ as a familiar surface, the graph of $f(t_1,t_2) = t_1t_2$ minus two lines.

4.3. **Computing the ideal of $X_{\mathcal{A}}$.** A variety is not just a set of points in projective space. An algebraic variety is a set that arises as the set of all points that are solutions to some finite set of polynomial equations. In this section we will learn how to compute the ideal of the variety $X_{\mathcal{A}}$.

To do this, let's think carefully about what it means for a polynomial on the ambient space to vanish at a point $\phi_{\mathcal{A}}(\mathbf{t})$. If $f(x_1,\ldots,x_n)$ is a polynomial, then we need to compute $f(\phi_{\mathcal{A}}(\mathbf{t}))$. This is done monomial-by-monomial. So, let's focus on what it really means to substitute $\phi_{\mathcal{A}}(\mathbf{t})$ into a monomial $\mathbf{x}^{\mathbf{u}}$. This is best done in an example. We will see that if $f = \mathbf{x}^{\mathbf{u}}$, the $f(\phi_{\mathcal{A}}(\mathbf{t})) = \mathbf{t}^{A\mathbf{u}}$ where A is the $d \times n$ matrix whose columns are the vectors $\mathbf{a}_1,\ldots,\mathbf{a}_n$.

Example 4.10. Let $\mathcal{A} = \{(1,0),(0,1),(1,1)\}$ so that

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Suppose that $f(x_1,x_2,x_3) = x_3 - x_1x_2 = \mathbf{x}^{(0,0,1)} - \mathbf{x}^{(1,1,0)}$. Then

$$\begin{aligned} f(\phi_{\mathcal{A}}(\mathbf{t})) =& f(\mathbf{t}^{(1,0)},\mathbf{t}^{(0,1)},\mathbf{t}^{(1,1)}) \\ =& \mathbf{t}^{0\cdot(1,0)}\mathbf{t}^{0\cdot(0,1)}\mathbf{t}^{1\cdot(1,1)} - \mathbf{t}^{1\cdot(1,0)}\mathbf{t}^{1\cdot(0,1)}\mathbf{t}^{0\cdot(1,1)} \\ =& \mathbf{t}^{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} - \mathbf{t}^{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}} \end{aligned}$$

What we observe is that substituting a monomial map into a monomial $\mathbf{x}^{\mathbf{u}}$ is equivalent to applying a linear transformation to $\mathbf{u}$ using the matrix of the monomial map. From this it is clear that if we have a binomial $f(\mathbf{x}) = \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$, then $f(\phi_{\mathcal{A}}(\mathbf{t}))$ is identically zero exactly when $A\mathbf{u} = A\mathbf{v}$. In fact, we have the following theorem:

Theorem 4.11 (Lemma 4.1 in [3]). *Suppose $\mathcal{A} \subset \mathbb{Z}^d$ is a finite ordered set and that A is the associated $d \times n$ matrix. Let $I_{X_\mathcal{A}}$ be the ideal of the set $X_\mathcal{A}$. Then $I_{X_\mathcal{A}}$ is equal to the ideal*

$$I_\mathcal{A} := \langle \mathbf{x}^\mathbf{u} - \mathbf{x}^\mathbf{v} \mid \mathbf{u} - \mathbf{v} \in \ker A,\ \mathbf{u}, \mathbf{v} \in \mathbb{Z}^n_{\geq 0} \rangle.$$

Proof. We will show that $I_\mathcal{A}$ is contained in $I_{X_\mathcal{A}}$ with a computation. If $f \in I_\mathcal{A}$, then it is a linear combination (with polynomial coefficients) of elements of the form $\mathbf{x}^\mathbf{u} - \mathbf{x}^\mathbf{v}$ where $A\mathbf{u} = A\mathbf{v}$. For each of these binomials

$$\mathbf{t}^{A\mathbf{u}} - \mathbf{t}^{A\mathbf{v}} = 0$$

becuase $A\mathbf{u} = A\mathbf{v}$. Therefore, $f(\phi_\mathcal{A}(\mathbf{t})) = 0$ and $f \in I_{X_\mathcal{A}}$.

Next, we will show that every element of $I_{X_\mathcal{A}}$ is in $I_\mathcal{A}$. Let U be the set of all polynomials in $I_{X_\mathcal{A}}$ that are not in $I_\mathcal{A}$. For each $f \in U$, consider the lexicographic leading term of f, i.e., the term that would come first in the dictionary. If U is not empty, then there is some nonzero f in U whose leading term is smallest. (The set of exponent vectors is bounded below by $\mathbf{0}$ and is discrete.)

Since $f \in I_\mathcal{A}$, $f(\phi_\mathcal{A}(\mathbf{t})) = 0$. Without loss of generality, assume that the leading term of f is $\mathbf{x}^\mathbf{u}$, so that the coefficient of the leading term is 1. If we expand $f(\phi_\mathcal{A}(\mathbf{t}))$, we will see a term of the form $\mathbf{t}^{A\mathbf{u}}$. Since this term cancels when we simplify, there must be at least one term of the form $\gamma\mathbf{t}^{A\mathbf{v}}$ with $A\mathbf{u} = A\mathbf{v}$ in the expression. Now define $f' = f - (\mathbf{x}^\mathbf{u} - \mathbf{x}^\mathbf{v})$. Notice that the leading term of f' is lexicographically smaller than the leading term of f as we have gotten rid of our original leading term $\mathbf{x}^\mathbf{u}$ and as $\mathbf{x}^\mathbf{v}$ appeared with nonzero coefficient in f, we have not introduced any additional monomials. If f' is in U, then we have contradicted the minimality of our choice of f. If f' is not in U, then neither is f which is again a contradiction. Therefore, U must be empty and the claim is proved. □

Corollary 4.12 (Corollary 4.3 in [3]). *Assume the hypotheses of Theorem 4.11. Suppose that $\mathbf{u} \in \ker A$. Write $\mathbf{u} = \mathbf{u}^+ - \mathbf{u}^-$, where $\mathbf{u}^+, \mathbf{u}_- \in \mathbb{Z}^n_{\geq 0}$. Then $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}_-} \in I_{X_\mathcal{A}}$.*

Example 4.13. Let A be as in 5.3. The vector $\mathbf{u} = (-1, -1, 1)$ is in the kernel of A (and actually spans the kernel). We can write it as a difference of its positive and negative pieces: $\mathbf{u} = (0, 0, 1) - (1, 1, 0)$.

We will see in Exercise 1 that the binomials corresponding to a basis for the kernel of A do not necessarily generate all of the ideal $I_\mathcal{A}$. However, if the kernel is one-dimensional, then the ideal $I_\mathcal{A}$ is principal.

%beginclaim

4.4. **Exercises.**

(1) Consider the matrix
$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}.$$
   (a) Let $\mathbf{v}_1 = (1, -2, 1, 0), \mathbf{v}_2 = (1, -1, -1, 1), \mathbf{v}_3 = (0, 1, -2, 1)$ Show that $\mathbf{v}_1$ is in the span of $\mathbf{v}_2$ and $\mathbf{v}_3$.
   (b) Write down the binomials corresponding to $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.
   (c) Is $\mathbf{v}_3$ in the ideal generated by $\mathbf{v}_1$ and $\mathbf{v}_2$? Can you explain your answer and how it relates to part (a)?

(2) Consider the integer matrices
$$A_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 & 2 \end{pmatrix}$$
$$A_3 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}, A_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$
   (a) Let $\mathcal{A}_i$ be the set of columns of A_i. Write down $\phi_{\mathcal{A}_i}$ for each i. (You might want to write each map using $\mathbf{x}^{\mathbf{a}}$ notation as well as in the usual notation in which you write down each variable explicitly.)
   (b) If I_{A_i} is nonzero, find at least three elements in I_{A_i} by hand.

(3) Show that if B is an $n \times n$ integer matrix, then it has an inverse (with integer entries) if and only if $\det B = \pm 1$. We will denote the set of all $n \times n$ invertible integer matrices by $GL_n(\mathbb{Z})$.

5. Projective toric varieties

We have defined an affine toric variety as the closure of the image of a monomial map $\phi_{\mathcal{A}}$. Although we have a very nice description of points in $\operatorname{Im} \phi_{\mathcal{A}}$, it is not clear that the points that we add when we take the closure of this set will have nice properties.

Let us revisit Example 1.1 again.

Example 5.1 (Parabola)**.** Recall that a set that is closed in the Zariski topology is also closed in the usual topology. Therefore, the Zariski closure of the image of the map $\phi_{\mathcal{A}}(t) = (t, t^2)$ must contain its limit points in the usual sense. We can see that as $t \to 0$, $\phi(t) \to (0, 0)$. We do not get a limit point when we send to t to ∞.

However, if we think of our parabola in $\mathbb{A}^2$ as a parabola sitting inside of an affine open patch of $\mathbb{P}^2$, then we will be able to see the limits at infinity by passing to other affine opens. For example, consider the map $\psi_{\mathcal{A}} : (\mathbb{C}^*)^2 \to \mathbb{P}^2$ given by $\psi_{\mathcal{A}}(t) = [1 : t : t^2]$. If $t \neq 0$ then $[1 : t : t^2] = [\frac{1}{t^2} : \frac{1}{t} : 1]$. So, we can see that the limit as $t \to \infty$ is $[0 : 0 : 1]$.

The closure of the image of $\psi_{\mathcal{A}}$ is defined by the single equation $xz = y^2$. When x is nonzero, we can multiply by a nonzero scalar so that $x = 1$. In this affine open patch of $\mathbb{P}^2$, the equation reduces to $z = y^2$ and the limit point in this patch is $[1 : 0 : 0]$. Similarly, when z is nonzero, we get the equation $x = y^2$ with limit point $[0 : 0 : 1]$. When y is nonzero, then both x and z must also be nonzero if $xz = y^2$. Therefore, we see that the closure of the image of $\psi_{\mathcal{A}}$ contains precisely two limit points not in $\operatorname{Im} \psi_{\mathcal{A}}$.

More generally, giiven a finite set of lattice points $\mathcal{P} = \{\mathbf{m}_0, \dots, \mathbf{m}_n\}$, define $\psi_{\mathcal{P}} : (\mathbb{C}^*)^2 \to \mathbb{P}^n$ by $\psi_{\mathcal{P}}(\mathbf{t}) = [\mathbf{t}^{\mathbf{m}_0} : \cdots : \mathbf{t}^{\mathbf{m}_n}]$. Note that since we do not allow any $t_i = 0$, this map is defined at every point of $(\mathbb{C}^*)^2$.

Definition 5.2. The projective toric variety $X_{\mathcal{P}}$ associated to a set of $n + 1$ lattice points $\mathcal{P}$ is the closure of the image of $\psi_{\mathcal{P}}$ in $\mathbb{P}^n$. The image of a permutation of the coordinates of $\psi_{\mathcal{P}}$ is isomorphic to $X_{\mathcal{P}}$ via a linear change of coordinates on $\mathbb{P}^n$. (See pg. 66 of [1], pg. 166 of [2], and pgs. 31 and 36 in [3].)

We would like to use Theorem 4.11 to compute the homogeneous ideal of $X_{\mathcal{P}}$, but as Example 5.3 will show, some modification is necessary.

Example 5.3. Suppose we consider the points $\mathcal{P} = \{(0,0), (1,0), (0,1), (1,1)\}$. Then $\psi_{\mathcal{P}} : (\mathbb{C}^*)^2 \to \mathbb{P}^3$ is given by $\psi_{\mathcal{P}}(t_1, t_2) = [1 : t_1 : t_2 : t_1 t_2]$. The matrix $\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ has a 1-dimensional kernel spanned by $(0, -1, -1, 1)$.

Let $f = z - xy$. Note that $f(1, t_1, t_2, t_1 t_2) = 0$, but this is not equivalent to the statement that $f(\psi_{\mathcal{P}}) = 0$. The reason that these two statements are not equivalent is that in projective space, the point $[1 : t_1 : t_2 : t_1 t_2] = \lambda[1 : t_1 : t_2 : t_1 t_2]$ for any nonzero λ.

For example, suppose that $\psi_{\mathcal{P}}(1, 2) = [1 : 1 : 2 : 2] = [3 : 3 : 6 : 6]$. We see that $f(1, 1, 2, 2) = 2 - 1 \cdot 2 = 0$, but that $f(3, 3, 6, 6) = 6 - 3 \cdot 6 = -12 \neq 0$. The problem, of course, is that f is not homogenous.

Theorem 5.4. *Let $\mathcal{P} = \{\mathbf{m}_0, \dots, \mathbf{m}_n\} \subset \mathbb{Z}^m$. Let $\mathbf{a}_i$ be the $d + 1$-dimensional vector with first coordinate 1 and last d coordinates given by $\mathbf{m}_i$. If $\mathcal{A} = \{\mathbf{a}_0, \dots, \mathbf{a}_n\}$, then $I_{\mathcal{A}}$ is the ideal of $X_{\mathcal{P}}$.*

Proof. A homogeneous polynomial f is in the ideal of $X_{\mathcal{P}}$ if and only if $f(\psi_{\mathcal{P}}(\mathbf{t}) = 0$ for every $\mathbf{t} \in (\mathbb{C}^*)^d$. A homogeneous polynomial vanishes at $\psi_{\mathcal{P}}(\mathbf{t})$ if and only if it vanishes at $\lambda \psi_{\mathcal{P}}(\mathbf{t})$ for all nonzero λ.

From Theorem 4.11 we know that $I_{\mathcal{A}}$ consists of all polynomials $f \in \mathbb{C}[x_0, \ldots, x_n]$ such that $f(\phi_{\mathcal{A}}(t_0, \ldots, t_d)) = 0$ for all $(t_0, \ldots, t_d) \in (\mathbb{C}^*)^{d+1}$. As $f(\phi_{\mathcal{A}}(t_0, \ldots, t_d)) = ft_0(\psi_{\mathcal{P}}(t_1, \ldots, t_d))$, we are done. □

Remark 5.5. If $\mathcal{A}$ is as in Theorem 5.4, then the corresponding $(d+1) \times n$ matrix A has top row equal to the all 1's vector. This imples that any $\mathbf{u}$ in the kernel of A has the property that the sum of its coordinates is zero. Equivalently, if we write $\mathbf{u} = \mathbf{u}^+\mathbf{u}^-$, then the sum of the coordinates of $\mathbf{u}^+$ equals the sum of the coordinates of $\mathbf{u}^-$. This in turn implies that $\deg \mathbf{x}^{\mathbf{u}^+} = \deg \mathbf{x}^{\mathbf{u}^-}$ and hence that $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-}$ is homogeneous.

Example 5.6. Let $\mathcal{P} = \{0, 1, 2, 3\} \subset \mathbb{Z}$. The corresponding matrix A is

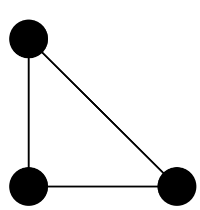
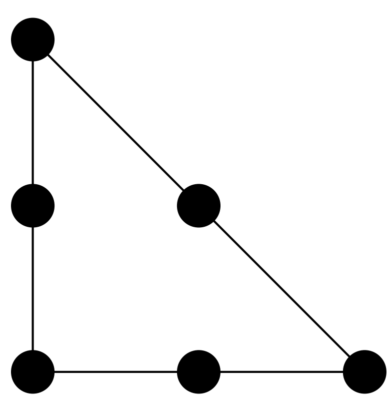
$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}.$$

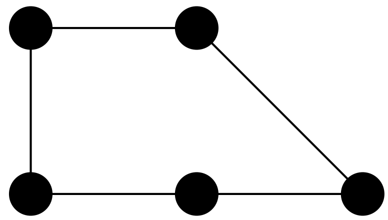
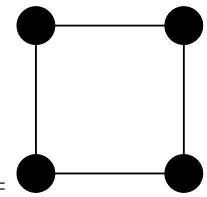
In Exercise 1 you showed that $\mathbf{v}_1 = (1, -2, 1, 0), \mathbf{v}_2 = (1, -1, -1, 1), \mathbf{v}_3 = (0, 1, -2, 1)$ are in the kernel of A. Note that the sum of the coordinates of each of these vectors is zero. Equivalently, the binomials $wy - x^2, wz - xy, xz - y^2$ are ll homogeneous of degree 2.

Typically, the sets $\mathcal{P}$ arise from lattice polytopes.

Notation 5.7. Suppose that $P \subset \mathbb{R}^d$ is a lattice polytope. Let $\mathcal{P} = P \cap \mathbb{Z}^d$. We will abuse notation and write ψ_P in place of $\psi_{\mathcal{P}}$ and X_P for the variety $X_{\mathcal{P}}$.

Example 5.8. The four lattice polygons below correspond to projective toric varieties via the construction described above.

$P_1 =$ , $P_2 =$ 

$P_3 =$  , $P_4 =$  ,

The corresponding maps are $\psi_{P_1}(t_1, t_2) = [1 : t_1 : t_2]$, $\psi_{P_2}(t_1, t_2) = [1 : t_1 : t_2 : t_1t_2 : t_1^2 : t_2^2]$, $\psi_{P_3}(t_1, t_2) = [1 : t_1 : t_2 : t_1t_2 : t_1^2]$, $\psi_{P_4}(t_1, t_2) = [1 : t_1 : t_2 : t_1t_2]$. We can compute their ideals using the following integer matrices.

$$A_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 & 2 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}, A_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

We will revisit these example frequently throughout the text to see how their toric realizations are related to more classical descriptions and to explore how they are related to each other.

6. Limit points of projective toric varieties

In this chapter we will study the action of the *torus* $(\mathbb{C}^*)^d$ on a variety X_P and compute the orbits of the torus action. Restricting to the case of surfaces for simplicity, we will see that this action helps us to decompose the limit points of X_P into pieces that fit together in a nice way.

6.1. **The torus action.**

Recall the following definitions from group theory.

Definition 6.1. A group G *acts* on a set S if we have defined $g \cdot s \in S$ for every $g \in G$ and $s \in S$ so that

1. $e \cdot s = s$, where $e \in G$ is the identity element.
2. $(gh) \cdot s = g \cdot (h \cdot s))$.

The action of a group G on a set S defines an equivalence relation on S with $s \equiv t$ if there exists some $g \in G$ so that $g \cdot s = t$. Given an element $s \in S$, the *orbit* of s is

$$\{t \in S \mid \exists g \in G \textit{ with } t = g \cdot s\}.$$

If s and t are in the same orbit, we write $s \equiv t$.

Definition 6.2. Let $T = (\mathbb{C}^*)^d$ denote the d-dimensional *algebraic torus*. The set T is a group under coordinatewise multiplication.

Example 6.3. If $T = (\mathbb{C}^*)^2$, and $(s_1, s_2), (t_1, t_2) \in T$, then $(s_1, s_2)(t_1, t_2) = (s_1t_1, s_2t_2)$. We will often use our monomial notation convention for elements of T, so that $\mathbf{s}$ denotes (s_1, s_2) and $\mathbf{st} = (s_1t_1, s_2t_2)$.The identity is $(1, 1)$. The inverse of (s_1, s_2) is $(\frac{1}{s_1}, \frac{1}{s_2})$, which is defined as neither s_1 nor s_2 may be zero.

Definition 6.4. Given finite set of lattice points $\mathcal{A} \subset \mathbb{R}^d$, we get an action of $T = (\mathbb{C}^*)^d$ on the image of $\phi_{\mathcal{A}}$. If $\mathbf{g} \in T$, define $\mathbf{g} \cdot \mathbf{t}^{A_i}$ by $\mathbf{g}^{a_i} \cdot \mathbf{t}^{A_i}$.

Note that $\mathbf{g} \cdot \phi_{\mathcal{A}}(\mathbf{t}) = \phi_{\mathcal{A}}(g_1t_1, \ldots, g_dt_d)$. This shows that T leaves the image of $\phi_{\mathcal{A}}$ fixed.

Theorem 6.5. *The action of T on* $\operatorname{Im} \phi_{\mathcal{A}}$ *is transitive.*

Proof. Suppose that $\phi_{\mathcal{A}}(\mathbf{s})$ and $\phi_{\mathcal{A}}(\mathbf{t})$ are two points in the image of $\phi_{\mathcal{A}}$. As T is a group, there is an element $\mathbf{g}$ such that $\mathbf{gs} = \mathbf{t}$. For this $\mathbf{g}$, we have $\mathbf{g} \cdot \phi_{\mathcal{A}}(\mathbf{s}) = \phi_{\mathcal{A}}(\mathbf{gs}) = \phi_{\mathcal{A}}(\mathbf{t})$. □

Let's look at some examples.

Example 6.6 (The triangle P_1)**.** The set $P_1 \cap \mathbb{Z}^2 = \{(0, 0), (1, 0), (0, 1)\}$. These lattice vectors give the map $\phi_{P_1} : (\mathbb{C}^*)^2 \to \mathbb{P}^2$ given by $\mathbf{t} \mapsto [1 : t_1 : t_2]$. The torus action is $\mathbf{g} \cdot [1 : t_1 : t_2] = [1 : g_1t_1 : g_2t_2]$.

Notice that the point $[1 : 1 : 1]$ is in the image of ψ_{P_1}. We'll determine the orbit of the point $[1 : 1 : 1]$ by computing $\mathbf{g} \cdot [1 : 1 : 1] = [1 : g_1 : g_2]$. We see that $\{[1 : g_1 : g_2] \mid \mathbf{g} \in T\}$ is exactly the image of ψ_{P_1}. Therefore, the orbit of $[1 : 1 : 1]$ is $\operatorname{Im} \psi_{P_1}$.

It follows from the definition that any group action is transitive on each orbit. We can check that the action is transitive on the image of ψ_{P_1} directly. Indeed, suppose that $[1 : t_1 : t_2]$ and $[1 : s_1 : s_2]$ are two points with $\mathbf{t}, \mathbf{s} \in (\mathbb{C}^*)^2$. If we set $g_i = \frac{s_i}{t_i}$, then $\mathbf{g} \cdot [1 : t_1 : t_2] = [1 : s_1 : s_2]$. Observe that the orbit of $[1 : 1 : 1]$ is exactly the set of points in $\mathbb{P}^2$ with no coordinate equal to zero since we are just multiplying the second two coordinates of $\mathbb{P}^2$ by nonzero elements of $\mathbb{C}^*$. It follows that the orbits of this action on $\mathbb{P}^2$ must correspond to sets of points with fixed coordinates set to 0.

We can give a list of representatives for each orbit: $\{[1 : 1 : 1], [1 : 1 : 0], [1 : 0 : 1], [0 : 1 : 1], [1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]\}$. The closure of the first orbit is all of $\mathbb{P}^2$, the closures of the next three orbits are the three coordinate axes. The last three points are fixed by the torus action, and each is the unique element in its orbit.

Example 6.7 (The square P_4)**.** As above, the orbit of $[1:1:1:1]$ is all of $\operatorname{Im}\psi_{P_4}$. Each of the four points $[1:0:0:0], [0:1:0:0], [0:0:1:0], [0:0:0:1]$ is fixed because multiplying their coordinates by nonzero coordinates leaves them unchanged and so each is its own orbit.

There are four other orbits represented by $[1:1:0:0], [1:0:1:0], [0:1:0:1]$, and $[0:0:1:1]$. We can see that these orbits are 1-dimensional. Indeed, $\mathbf{g}\cdot[1:1:0:0] = [1:g_1:0:0]$, is an affine line minus a point as are $\mathbf{g}\cdot[1:0:1:0] = [1:0:g_2:0]$, $\mathbf{g}\cdot[0:1:0:1] = [0:g_1:0:g_1g_2] = [0:1:0:g_2]$ and $\mathbf{g}\cdot[0:0:1:1] = [0:0:g_2:g_1g_2] = [0:0:1:g_1]$.

With the information we have seen so far it is not clear that we have found all of the orbits of the action of T and X_{P_4}. In fact, it is not even clear that any of the orbit representatives other than $[1:1:1:1]$ are in the toric variety. However, these points are limit points of the image of ψ_{P_4}, and we will start to see how to compute them in the next section.

To generalize what we have seen in the examples above, notice that there are as many torus-fixed points as vertices in each polygon, and as many torus-invariant curves as edges. (Of course, for surfaces, the number or edges is always equal to the number of vertices, but associating points to vertices, etc is what is correct in higher dimensions.) The decomposition of a polygon into its 2-dimensional face, edges, and vertices corresponds to the orbit-closure decomposition of the corresponding toric variety.

6.2. **Limits.** In this section, we answer the question: what do we get when we take the closure of the image of our embeddings?

6.2.1. *Affine varieties.* We can use the torus action to find the limit points of $X_{\mathcal{A}}$. We will do this by parameterizing curves in $(\mathbb{C}^*)^2$ and taking their limits as the parameter goes to zero. The limit will be a point whose coordinates are all zero or 1. The orbit of such a point will either be the image of the torus, a dense subset of a torus-invariant curve, or a torus-fixed point. The 1-dimensional torus-orbits together with the torus-fixed points are the limit points that we add in when we take the closure of the image of $\phi_{\mathcal{A}}$.

The curves in $(\mathbb{C}^*)^2$ that we will use to find limit points have a special form.

Definition 6.8 (pg. 37 in [1])**.** An integer vector $\mathbf{v}\in\mathbb{Z}^2$ corresponds to a map $\lambda_{\mathbf{v}}:\mathbb{C}^*\to(\mathbb{C}^*)^2$ given by

$$\lambda_{\mathbf{v}}(z) = (z^{v_1}, z^{v_2}).$$

The image of $\lambda_{\mathbf{v}}$ is a subgroup of $(\mathbb{C}^*)^2$ and we call it a *1-parameter subgroup.* If $\mathbf{v} \neq \mathbf{0}$, then $\operatorname{Im}\lambda_{\mathbf{v}}$ is a curve. (Note that $\lambda_{\mathbf{0}}(z)(1,1)$.)

Example 6.9. Suppose we start with $\mathcal{A} = \{(1,0),(0,1)\}$ so that $\phi_{\mathcal{A}}(\mathbf{t}) = (t_1, t_2)$. We want to consider

$$\lim_{z\to 0} \phi_{\mathcal{A}}(\lambda_{\mathbf{v}}(z))$$

for various lattice vectors $\mathbf{v}$. If $\mathbf{v} = (v_1, v_2)$ then we get

$$\lim_{z\to 0} \phi_{\mathcal{A}}(\lambda_{\mathbf{v}}(z)) = \lim_{z\to 0}(z^{v_1}, z^{v_2}).$$

Let us see what is going on when $\mathbf{v} = (1,2)$. Then $\lambda_{(1,2)}(z) = (z, z^2)$ is a parabola in the plane. We want to see what happens to the points $\phi_{\mathcal{A}}(z,z^2) = (z,z^2)$ on our surface as $z \to 0$. In this case it is easy to see that the limit is $(0,0)$. If we had picked $\mathbf{v} = (-1,0)$ then $\lim_{z\to 0}\phi_{\mathcal{A}}(\lambda_{\mathbf{v}}(z)) = \lim_{z\to 0}(\frac{1}{z}, 1)$ which does not exist.

In fact, it is easy to see that the limit exists if and only if $v_1, v_2 \geq 0$, i.e., that $\mathbf{v}$ is in the first quadrant (including the axes). Moreover, we can see that there are four different possible limits, according to the cases $v_1 = v_2 = 0$, $v_1 > 0, v_2 = 0$, $v_1 = 0, v_2 > 0$, and $v_1, v_2 > 0$. Picking a representative for each possibility we can summarize the results with the table below.

$\mathbf{v}$	$\phi_{\mathcal{A}}(\lambda_{\mathbf{v}}(z))$	$\lim_{z\to 0}\phi_{\mathcal{A}}(\lambda_{\mathbf{v}}(z))$
$(0,0)$	$(1,1)$	$(1,1)$
$(1,0)$	$(z,1)$	$(0,1)$
$(0,1)$	$(1,z)$	$(1,0)$
$(1,1)$	(z,z)	$(0,0)$

Let us look at another example.

Example 6.10. Suppose that $\mathcal{A} = \{(-1,0),(-1,1)\}$. For which $\mathbf{v}$ does $\lim_{z\to 0}\phi_{\mathcal{A}}(\lambda_{\mathbf{v}}(z))$ exist? Notice that $\phi_{\mathcal{A}}(\lambda_{\mathbf{v}}(z)) = \phi_{\mathcal{A}}(z^{v_1}, z^{v_2}) = (z^{-v_1}, z^{-v_1+v_2})$. Therefore, the limit as $z \to 0$ exists if and only if $-v_1, -v_1+v_2 \geq 0$. Geometrically, this means that (v_1, v_2) lies in the intersection of the halfplanes $v_1 \leq 0$ and $v_1 \leq v_2$ depicted below. Again, there are four possibilities: $v_1 = v_2 = 0$, $v_1 < 0, v_2 = v_1$, $v_1 = 0, v_2 > 0$, and $v_1 < 0, v_2 > v_1$. Choosing representatives of each case we have

$\mathbf{v}$	$\lambda_{\mathbf{v}}(z)$	$\phi_{\mathcal{A}}(\lambda_{\mathbf{v}}(z))$	$\lim_{z\to 0}\phi_{\mathcal{A}}(\lambda_{\mathbf{v}}(z))$
$(0,0)$	$(1,1)$	$(1,1)$	$(1,1)$
$(-1,-1)$	$(\frac{1}{z},\frac{1}{z})$	$(z,1)$	$(0,1)$
$(0,1)$	$(1,z)$	$(1,z)$	$(1,0)$
$(-1,1)$	$(\frac{1}{z},z)$	(z,z)	$(0,0)$

In general, the lattice points in $\mathcal{A}$ generate a cone. The limit $\lim_{z\to 0}\phi_{\mathcal{A}}(\lambda_{\mathbf{v}}(z))$ exists if and only if $\mathbf{v}$ is in the cone *dual* to the cone generated by $\mathcal{A}$. This is just the set of vectors whos dot produce with every vector in cone $\mathcal{A}$ is nonnegative. There will be a special limit point for each face of the dual of cone $\mathcal{A}$. If cone $\mathcal{A}$ is 2-dimensional then it has precisely 4 faces: one 2-dimensional face, two edges, and the cone point which is a zero-dimensional face. The limit points form a set of representatives for the orbits of the action of T on $X_{\mathcal{A}}$. In the 2-dimensional case, there is one dense orbit, one fixed point, and two orbits that are curves.

6.2.2. *Projective varieties.*

Fact 6.11. If P is a lattice polygon, then X_P is the union of the image of ψ_P together with a projective torus-invariant curve for each edge of P. Two edges meet in a vertex in P if and only if the corresponding curves in X_P meet in a point that is fixed by the torus action.

Example 6.12 (The triangle P_1)**.** We have $(t_1, t_2) \in (\mathbb{C}^*)^2$ mapping to $[1 : t_1 : t_2]$.

$\mathbf{v}$	$\lambda_{\mathbf{v}}(z)$	$\phi_{P_1}(\lambda_{\mathbf{v}}(z))$	$\lim_{z\to 0}\phi_{P_1}(\lambda_{\mathbf{v}}(z))$
$(1,1)$	(z,z)	$[1:z:z]$	$[1:0:0]$
$(1,-1)$	$(z,\frac{1}{z})$	$[1:z:\frac{1}{z}]$	$[0:0:1]$
$(-1,1)$	$(\frac{1}{z},z)$	$[1:\frac{1}{z}:z]$	$[0:1:0]$
$(1,0)$	$(z,1)$	$[1:z:1]$	$[1:0:1]$
$(0,1)$	$(1,z)$	$[1:1:z]$	$[1:1:0]$
$(-1,-1)$	$(\frac{1}{z},\frac{1}{z})$	$[1:\frac{1}{z}:\frac{1}{z}]$	$[0:1:1]$

Notice that the first three limit points are fixed by the torus action. The last three points have orbits equal to $\mathbb{C}^*$. We obtain the closures of these orbits by adding in the appropriate fixed points to get projective lines.

Example 6.13 (The big triangle P_2)**.** The lattice points in P_2 give the map $\phi_{P_2} : (\mathbb{C}^*)^2 \to \mathbb{P}^5$ sending

$$(t_1, t_2) \mapsto [1 : t_1 : t_2 : t_1t_2 : t_1^2 : t_2^2].$$

Again, we make a table

$\mathbf{v}$	$\lambda_{\mathbf{v}}(z)$	$\phi_{P_2}(\lambda_{\mathbf{v}}(z))$	$\lim_{z\to 0}\phi_{P_2}(\lambda_{\mathbf{v}}(z))$
$(1,1)$	(z,z)	$[1:z:z:z^2:z^2:z^2]$	$[1:0:0:0:0:0]$
$(1,-1)$	$(z,\frac{1}{z})$	$[1:z:\frac{1}{z}:1:z^2:\frac{1}{z^2}]$	$[0:0:0:0:0:1]$
$(-1,1)$	$(\frac{1}{z},z)$	$[1:\frac{1}{z}:z:1:\frac{1}{z^2}:z^2]$	$[0:0:0:0:1:0]$
$(1,0)$	$(z,1)$	$[1:z:1:z:z^2:1]$	$[1:0:1:0:0:1]$
$(0,1)$	$(1,z)$	$[1:1:z:z:1:z^2]$	$[1:1:0:0:1:0]$
$(-1,-1)$	$(\frac{1}{z},\frac{1}{z})$	$[1:\frac{1}{z}:\frac{1}{z}:\frac{1}{z^2}:\frac{1}{z^2}:\frac{1}{z^2}]$	$[0:0:0:1:1:1]$

The first three points correspond to fixed points. What happens to the last three under the torus action? We have

$$\mathbf{g}\cdot[1:0:1:0:0:1] = [1:0:g_2:0:0:g_2^2],$$

$$\mathbf{g}\cdot[1:1:0:0:1:0] = [1:g_1:0:0:g_1^2:0],$$

$$\mathbf{g}\cdot[0:0:0:1:1:1] = [0:0:0:g_1g_2:g_1^2:g_2^2].$$

What we see then, is that the closure of the image of ϕ_{P_2} contains three plane conics. In fact, what we are seeing is an embedding of $\mathbb{P}^2$ in which the three coordinate lines that we found earlier are all mapped to conics.

Note that the three edges in P_1 all had length one measured along the lattice, and in the limit we got three lines. The edges of P_2 all have lattice length 2, and in the limit we got three degree 2 curves. We see that P_3 has 3 edges of length 1 and one edge of length 2, so we expect to see limit points falling along three lines and one plane conic.

Example 6.14 (The trapezoid P_3)**.** For the trapezoid, we get the map $(t_1,t_2)\mapsto[1:t_1:t_2:t_1t_2:t_1^2]$. We find the torus invariant curves:

$\mathbf{v}$	$\lambda_{\mathbf{v}}(z)$	$\phi_{P_3}(\lambda_{\mathbf{v}}(z))$	$\lim_{z\to 0}\phi_{P_3}(\lambda_{\mathbf{v}}(z))=p$	$\mathbf{g}\cdot p$
$(1,0)$	$(z,1)$	$[1:z:1:z:z^2]$	$[1:0:1:0:0]$	$[1:0:g_2:0:0]$
$(0,1)$	$(1,z)$	$[1:1:z:z:1]$	$[1:1:0:0:1]$	$[1:g_1:0:0:g_1^2]$
$(-1,-1)$	$(\frac{1}{z},\frac{1}{z})$	$[1:\frac{1}{z}:\frac{1}{z}:\frac{1}{z^2}:\frac{1}{z^2}]$	$[0:0:0:1:1]$	$[0:0:0:g_2:g_1]$
$(0,-1)$	$(1,\frac{1}{z})$	$[1:1:\frac{1}{z}:\frac{1}{z}:1]$	$[0:0:1:1:0]$	$[0:0:1:g_1:0]$

We see that the orbit closures will be three projective lines and one plane conic.

Example 6.15 (The square P_4)**.** We have the map $\phi_{P_4} : (\mathbb{C}^*)^2 \to \mathbb{P}^3$ sending $\mathbf{t} \mapsto [1 : t_1 : t_2 : t_1t_2]$.

$\mathbf{v}$	$\lambda_{\mathbf{v}}(z)$	$\phi_{P_4}(\lambda_{\mathbf{v}}(z))$	$\lim_{z\to 0}\phi_{P_4}(\lambda_{\mathbf{v}}(z))$	$\mathbf{g}\cdot p$
$(1,0)$	$(z,1)$	$[1 : z : 1 : z]$	$[1 : 0 : 1 : 0]$	$[1 : 0 : g_2 : 0]$
$(0,1)$	$(1,z)$	$[1 : 1 : z : z]$	$[1 : 1 : 0 : 0]$	$[1 : g_1 : 0 : 0]$
$(-1,0)$	$(\frac{1}{z},1)$	$[1 : \frac{1}{z} : 1 : \frac{1}{z}]$	$[0 : 1 : 0 : 1]$	$[0 : 1 : 0 : g_2]$
$(0,-1)$	$(1,\frac{1}{z})$	$[1 : 1 : \frac{1}{z} : \frac{1}{z}]$	$[0 : 0 : 1 : 1]$	$[0 : 0 : 1 : g_1]$

We see that the orbit closures are 4 lines.

Right now the choices of $\mathbf{v}$ are a bit mysterious, but we will see later how to make sense of them.

6.3. **Exercises.**

(1) Suppose that a group G acts on a set S and that $s, t \in S$. Show that if there exists a $g \in G$ such that $g \cdot s = t$, then there is an $h \in G$ such that $h \cdot t = s$.

(2) Prove that $T = (\mathbb{C}^*)^d$ is a group.

(3) Let P be as in Definition 6.4. Prove that the action defined there extends to projective space: in other words, that $\mathbf{g} \cdot [x_0 : \cdots : x_n] = [\mathbf{g}^{\mathbf{m}_0} x_0 : \cdots : \mathbf{g}^{\mathbf{m}_n} x_n]$ is well-defined.

(4) Show that if $\mathbf{v} \in \mathbb{Z}^2$, then the set $\{(z^{v_1}, z^{v_2})\}$ is a subgroup of T.

(5) Find the torus fixed points of X_{P_i}, $i = 3, 4$.

(6) Substitute $t_1 = \frac{x_1}{x_0}$ and $t_2 = \frac{x_2}{x_0}$ into the map ϕ_{P_2}. Homogenize the map to show that this allows us to identify X_{P_2} with the image of a map $\nu_2 : \mathbb{P}^2 \to X_{P_2}$. The map ν_2 is the quadratic *Veronese* emebdding of $\mathbb{P}^2$ into $\mathbb{P}^5$. Show that ν_2 is injective.

    More generally, we have d-uple Veronese embeddings $\nu_d : \mathbb{P}^n \to \mathbb{P}^N$ whose coordinates may be given by a basis for the monomials in the homogeneous coordinates on $\mathbb{P}^n$ of degree d. What is N? Can you show that all of these maps are injective?

(7) We can define maps ψ_i from $\mathbb{P}^2$ to X_{P_i} where $i = 3, 4$ by deleting the appropriate coordinates from ν_2. These maps are not defined on all of $\mathbb{P}^2$. Determine the points where each map is undefined.

(8) Consider the closure of the maps ψ_i above. Can you identify the points that we add when we take the closure? Are the maps injective? If not, can you determine which points map to the same image?

(9) Notice that the map ϕ_{P_3} can be decomposed as

$$[1 : t_1 : t_2 : t_1t_2 : t_1^2] = [1 : t_1 : 0 : 0 : t_1^2] + t_2[0 : 0 : 1 : t_1 : 0].$$

Substitute $t_2 = \frac{y_1}{y_0}$ into each summand. Convince yourself that X_{P_3} has the following description: Map $\mathbb{P}^1$ into $\mathbb{P}^4$ simultaneously as a plane conic and a line where the line and plane do not meet. Then construct a surface by taking the union of all lines connecting the image of $p \in \mathbb{P}^1$ on the conic to its image on the line.

(10) Substitute $t_1 = \frac{x_1}{x_0}$ and $t_2 = \frac{y_1}{y_0}$ into the map ϕ_{P_4} and dehomogenize the map. Show that this gives an injective map from $\mathbb{P}^1 \times \mathbb{P}^1 = \{([x_0 : x_1], [y_0 : y_1]) \mid [x_0 : x_1] \in \mathbb{P}^1,\ [y_0 : y_1]) \in \mathbb{P}^1\}$ to X_{P_4}. This is the *Segre embedding* of $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$. Can you give a natural description of the 1-dimensional orbit closures in X_{P_4} in terms of $\mathbb{P}^1 \times \mathbb{P}^1$?

In general, we have the Segre embeddings $\mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^N$ whose coordinates are all products of homgeneous coordinates for $\mathbb{P}^m$ with homogeneous coordinates for $\mathbb{P}^n$. What is N? Can you show that the Segre embeddings are injective?

7. Coordinate rings of affine toric varieties

In this section we will explore the connection between the set $\mathcal{A}$ and the monomials that span the coordinate ring of $X_{\mathcal{A}}$.

We can view the coordinate ring of affine space as a model for the setup for toric varieties more generally.

Example 7.1. The set $\mathcal{A} = \{1, 2\}$ gives rise to the familiar parameterization of the parabola in the plane $\phi_{\mathcal{A}}(t) = (t, t^2)$. We can think of this map as giving a ring homomorphism $\phi_{\mathcal{A}}^{\#} : \mathbb{C}[x, y] \to \mathbb{C}[t]$ where $\phi_{\mathcal{A}}^{\#}(x) = t$ and $\phi_{\mathcal{A}}^{\#}(x) = t^2$. Theorem 4.11 shows that $I_{\mathcal{A}} = \langle y - x^2 \rangle$ is the kernel of this homomorphism. What is its image? As t is in the image of $\phi_{\mathcal{A}}^{\#}$, so is any polynomial in t. So we see that $\operatorname{Im} \phi_{\mathcal{A}}^{\#} = \mathbb{C}[t]$. By the First Isomorphism Theorem for rings $\mathbb{C}[x, y]/\langle y - x^2 \rangle \cong \mathbb{C}[t]$.

Example 7.2. Let $\mathcal{A} = \{(1, 0), (0, 1)\}$. Then $\phi_{\mathcal{A}} : (\mathbb{C}^*)^2 \to \mathbb{C}^2$ is given by $\phi_{\mathcal{A}} = (t_1, t_2)$. The map $\phi_{\mathcal{A}}$ gives rise to a ring homomorphism $\phi_{\mathcal{A}}^{\#} : \mathbb{C}[x_1, x_2] \to \mathbb{C}[t_1, t_2]$ as follows. Define $\phi_{\mathcal{A}}^{\#}(x_i) = t_i$ on the generators of $\mathbb{C}[x_1, x_2]$ and define $\phi_{\mathcal{A}}^{\#}(\alpha) = \alpha$ for each $\alpha \in \mathbb{C}$. If $\phi_{\mathcal{A}}^{\#}$ is a ring homomorphism, this information completely determines the map. For each monomial $x_1^{a_1} x_2^{a_2}$ we have

$$\phi_{\mathcal{A}}^{\#}(x_1^{a_1} x_2^{a_2}) = \phi_{\mathcal{A}}^{\#}(x_1)^{a_1} \phi_{\mathcal{A}}^{\#}(x_2)^{a_2} = t_1^{a_1} t_2^{a_2}$$

as $\phi_{\mathcal{A}}^{\#}$ preserves multiplication. Therefore, for a polynomial $f(\mathbf{x}) = \sum \alpha_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}$ we have $\phi_{\mathcal{A}}^{\#}(f(\mathbf{x})) = \sum \alpha_{\mathbf{a}} \mathbf{t}^{\mathbf{a}}$, and we see that the map is an isomorphism. Its image is spanned by the set of all monomials in t_1

and t_2. Note that the exponent vectors of these monomials form the set $\{(a,b) \mid a,b \in \mathbb{N}\}$. This is the intersection of the cone generated by $\mathcal{A}$ with the lattice $\mathbb{Z}^2$.

More generally, suppose we have $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{Z}^d$. The parameterization $\phi_{\mathcal{A}}$ gives rise to a ring homorphism $\phi_{\mathcal{A}}^{\#} : \mathbb{C}[\mathbf{x}] \to \mathbb{C}[\mathbf{t}]$ where $\phi_{\mathcal{A}}^{\#}(x_i) = \mathbf{t}^{\mathbf{a}_i}$. Theorem 4.11 implies that the kernel of $\phi_{\mathcal{A}}^{\#}$ is exactly $I_{\mathcal{A}}$. Thus, the First Isomorphism Theorem tells us that $\mathbb{C}[\mathbf{x}]/I_{\mathcal{A}}$, which is the affine coordinate ring of $X_{\mathcal{A}}$, isomorphic to $\operatorname{Im}\phi_{\mathcal{A}}^{\#}$.

From the description of $\phi_{\mathcal{A}}^{\#}$, we can see that $\operatorname{Im}\phi_{\mathcal{A}}^{\#}$ is spanned by monomials. We want to describe the exponent vectors of these monomials.

Definition 7.3. Given $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{Z}^d$ we define the *semigroup generated by* $\mathcal{A}$ to be $S_{\mathcal{A}} = \{r_1\mathbf{a}_1 + \cdots + r_n\mathbf{a}_n \mid r_i \in \mathbb{N}\}$.

Theorem 7.4. *The set of exponent vectors of monomials in* $\operatorname{Im}\phi_{\mathcal{A}}^{\#}$ *is* $S_{\mathcal{A}}$.

Proof. Since $\phi_{\mathcal{A}}^{\#}$ is a homomorphism, $\phi_{\mathcal{A}}^{\#}(x_1^{r_1}\cdots x_n^{r_n}) = \mathbf{t}^{r_1\mathbf{a}_1}\cdots\mathbf{t}^{r_n\mathbf{a}_n} = \mathbf{t}^{r_1\mathbf{a}_1+\cdots+r_n\mathbf{a}_n}$. Therefore, we can see that $S_{\mathcal{A}}$ has the form claimed above. □

Suppose that σ is the cone generated by $\mathcal{A}$. Let $S_\sigma = \sigma \cap \mathbb{Z}^d$. Then $S_{\mathcal{A}} \subseteq S_\sigma$. In general, $S_{\mathcal{A}}$ may not be equal to S_σ.

Example 7.5. Let $\mathcal{A} = \{(2,0),(0,2)\}$. Then $S_\sigma = \mathbb{N}^2$, but $S_{\mathcal{A}} = \{(a,b) \mid 2|a,\ d|b\}$.

In the following chapters we will restrict our attention to affine toric varieties such that $S_{\mathcal{A}} = S_\sigma$. This means that the semigroup generated by $\mathcal{A}$ is *saturated*, or that the affine coordinate ring of $X_{\mathcal{A}}$ is *normal.*

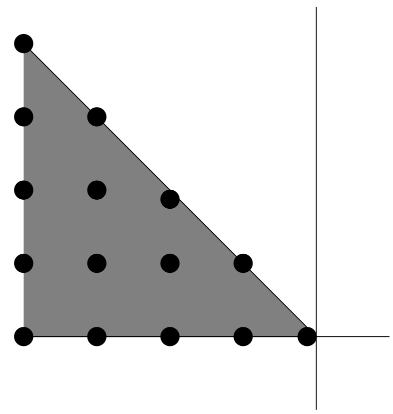
Example 7.6. Suppose we consider the set

$$\mathcal{A}' = \{(1,0),(0,1),(1,1),(2,0),(0,2)\}.$$

Then $S_{\mathcal{A}'} = S_{\mathcal{A}}$ where $\mathcal{A}$ was defined in Example 7.2. Here, $\phi_{\mathcal{A}'} : (\mathbb{C}^*)^2 \to \mathbb{C}^5$, and the kernel of $\phi_{\mathcal{A}}^{\#}$ is the ideal $I_{\mathcal{A}'} = \langle x_2^2 - x_5, x_1x_2 - x_3, x_1^2 - x_4\rangle$. Note that in the quotient ring $\mathbb{C}[\mathbf{x}]/I_{\mathcal{A}'}$, the coset $x_3 + I_{\mathcal{A}'}$ is equal to $x_1x_2 + I_{\mathcal{A}'}$ as $x_1x_2 - x_3 \in I_{\mathcal{A}'}$. In fact, each indeterminate x_3, x_4, x_5 can each be rewritten in terms of x_1 and x_2 modulo $I_{\mathcal{A}'}$. Thus, the quotient ring has a basis of monomials in x_1 and x_2. Indeed, by the First Isomorphism Theorem for rings, $\mathbb{C}[\mathbf{x}]/I_{\mathcal{A}'} \cong \mathbb{C}[\mathbf{t}]$ again.

In fact, if $\mathcal{B} \subset \mathbb{Z}^2$ and $S_{\mathcal{B}} = \mathbb{N}^2$, then the ring $\mathbb{C}[\mathbf{x}]/I_{\mathcal{B}}$ is isomorphic to a polynomial ring in two variables, which is the coordinate ring of affine 2-space. What we are seeing then, is that if we embedd $\mathbb{A}^2$ into another $\mathbb{A}^n$, then the ring $\mathbb{C}[x_1, \ldots, x_n]$ modulo the ideal of the image of $\mathbb{A}^2$ is isomorphic to the polynomial ring in two variables. Indeed each embedding of $\mathbb{A}^2$ into affine n-space gives us another realization of $\mathbb{C}[t_1, t_2]$ as the quotient ring of a polynomial ring.

Example 7.7. Suppose we let $\mathcal{B} = \{(-1, 0), (-1, 1)\}$. Again, we have a ring homomorphism $\phi_{\mathcal{B}}^{\#} : \mathbb{C}[x, y] \to \mathbb{C}[t_1, t_2]$ that is determined by $\phi_{\mathcal{B}}^{\#}(x) = \frac{1}{t_1}$ and $\phi_{\mathcal{B}}^{\#}(y) = \frac{t_2}{t_1}$. The image of $\phi_{\mathcal{B}}^{\#}$ is the subring of $\mathbb{C}[t_1.t_2, \frac{1}{t_1}, \frac{1}{t_2}]$ spanned by $\frac{1}{t_1}$ and $\frac{t_2}{t_1}$. The exponent vectors of monomials in $\operatorname{Im} \phi_{\mathcal{B}}^{\#}$ lie in the cone depicted below.



We can see that $\operatorname{Im} \phi_{\mathcal{B}}^{\#}$ is isomorphic to $\mathbb{C}[x, y]$ as the kernel of $B = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$ is 0, which implies that $I_{\mathcal{B}}$ is also zero.

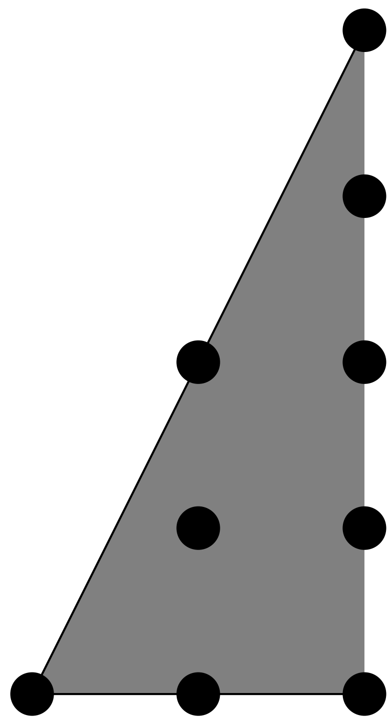
This isomorphism of rings also gives us an isomorphism of semigroups. The matrix B sends the standard basis vectors to $(-1, 0)$ and $(-1, 1)$. Since B is invertible, this map has an inverse.

Let $\sigma \subset \mathbb{R}^2$ be a rational convex polyhedral cone and $S_\sigma = \sigma \cap \mathbb{Z}^2$ be the semigroup of lattice points in the cone. If $X_{\mathcal{A}}$ is an arbitrary affine toric surface, the subring of $\mathbb{C}[t_1, t_2]$ spanned by the monomials with exponents in $S_{\mathcal{A}}$ can be thought of as the coordinate ring of $X_{\mathcal{A}}$ without reference to any embedding. A specific choice of generators of the semigroup $S_{\mathcal{A}}$ corresponds to a specific embedding

Example 7.8. Suppose now that we let $\mathcal{A} = \{(1, 0), (1, 1), (1, 2)\}$. Let $\sigma = \operatorname{cone} \mathcal{A}$. The matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

has kernel with basis $(1, -2, 1)$.



7.1. **Exercises.**

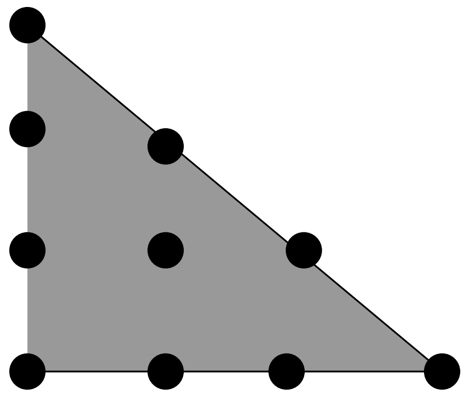
(1) Let $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\} \subset \mathbb{Z}^d$. We wish to define a ring homomorphism $\phi^{\#}_{\mathcal{A}} : \mathbb{C}[\mathbf{x}] \to \mathbb{C}[\mathbf{t}]$. Let $\phi^{\#}_{\mathcal{A}}(x_i) = \mathbf{t}^{\mathbf{a}}$ and $\phi^{\#}_{\mathcal{A}}$ be the identity on $\mathbb{C}$. If $\phi^{\#}_{\mathcal{A}}$ is a ring homomorphism, what is $\phi^{\#}_{\mathcal{A}}(\sum \alpha_{\mathbf{m}} \mathbf{t}^{\mathbf{m}})$?

(2) Consider the quotient ring $\mathbb{C}[\mathbf{x}]/I_{\mathcal{A}'}$ as in Example 7.6. Find a representative of the coset of $x_3x_4 - x_5^2$ using only x_1 and x_2.

8. Affine open covers and inner normal fans

Example 8.1. Let $Q_0 = \operatorname{conv}\{(0,0), (1,0), (0,1)\}$ be the standard simplex in the plane. We get the map $\psi_{Q_0}(\mathbf{t}) = [1 : t_1 : t_2]$ which maps to U_0. To see the affine variety $X_{Q_0} \cap U_0$, as above let $\mathcal{A}_0 = \{(1,0), (0,1)\}$. Then $\phi_{\mathcal{A}_0}(\mathbf{t}) = (t_1, t_2)$, and we see that $X_{\mathcal{A}_0} = \mathbb{C}^2$. In this case $\mathbb{C}[t_1, t_2] \cong \mathbb{C}[x, y]$ is just a polynomial ring. Moreover, note that the exponent vectors of monomials in $\mathbb{C}[t_1, t_2]$ are precisely the lattice vectors in the first quadrant.

Now consider $Q_1 = \operatorname{conv}\{(-1,0), (0,0), (-1,1)\}$, which is just a translation of Q_0. This gives rise to the map $\psi_{Q_1}(\mathbf{t}) = [\frac{1}{t_1} : 1 : \frac{t_2}{t_1}]$. Of course, $[\frac{1}{t_1} : 1 : \frac{t_2}{t_1}] = [1 : t_1 : t_2]$ if $t_1 \neq 0$, so then $X_{Q_0} = X_{Q_1}$ even though ψ_{Q_1} maps $(\mathbb{C}^*)^2$ to U_1. The affine variety that we see is again $\mathbb{C}^2$, although in this case $\mathcal{A}_1 = \{(-1,0),(-1,1)\}$. and $\mathbb{C}[\frac{1}{t_1},\frac{t_2}{t_1}] \cong \mathbb{C}[x,y]$ has exponent vectors contained in the cone below.



The affine toric variety gotten from translating the top vertex of the triangle to the origin can be analyzed in a smilar fashion.

We introduce language that will allow us to discuss the example above more precisely.

Definition 8.2. A finite set of vectors $\mathcal{A} = \{\mathbf{a}_1,\ldots,\mathbf{a}_n\} \subset \mathbb{R}^d$ generates a *convex polyhedral cone*

$$\operatorname{cone}\mathcal{A} = \{\lambda_1\mathbf{a}_1 + \cdots + \lambda_n\mathbf{a}_n \mid \lambda_i \in \mathbb{R}_{\geq 0}\}.$$

The dimension of a cone is the dimension of its linear span. A cone is *rational* if it can be generated by lattice vectors. We will simply write "cone" in these notes but will always mean a rational convex polyhedral cone. We say that the cone is *strongly convex* if it doesn't contain $\mathbf{v}$ and $-\mathbf{v}$ for any $\mathbf{v} \in \mathbb{R}^d$. (See Proposition 3 on pg. 14 of [1].)

Definition 8.3. If $\sigma \subset \mathbb{R}^d$ is a cone, we can define the ring $\mathbb{C}[\mathbf{t}^{\mathbf{a}} \mid \mathbf{a} \in \mathbb{Z}^d \cap \sigma]$. In other words, we have defined a ring that is generated by monomials whose exponent vectors lie in the cone σ.

Assumption 8.4. Assume that $\mathcal{A}$ generates the set $(\operatorname{cone}\mathcal{A}) \cap \mathbb{Z}^d$ as a semigroup.

Fact 8.5. Let $P \subset \mathbb{R}^d$ be a lattice polygon with $P \cap \mathbb{Z}^d = \{\mathbf{a}_0 = \mathbf{0},\ldots,\mathbf{a}_n\}$. If $\sigma = \operatorname{cone}\{\mathbf{a}_1,\ldots,\mathbf{a}_n\}$ and $A = (\mathbf{a}_1 \cdots \mathbf{a}_n)$, then $\mathbb{C}[\mathbf{t}^{\mathbf{a}_1},\ldots,\mathbf{t}^{\mathbf{a}_n}] \cong \mathbb{C}[x_1,\ldots,x_n]/I_{\mathcal{A}}$.

8.0.1. *Keeping track of cones: fans and duals.*

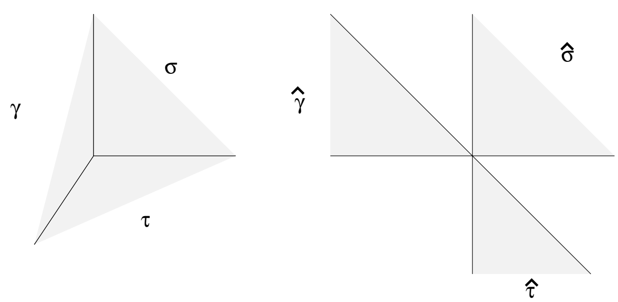
Example 8.6 (X_{P_3})**.** Suppose we try to draw the cone at each vertex of P_3. Notice that we can't draw all of them in the same picture without overlaps.

There is a *dual* picture that involves all of the same data, but gives us better intuition about how the different affine patches interact with each other.

Definition 8.7. If $\sigma \subset \mathbb{R}^d$ is a cone, then its *dual* is defined to be

$$\hat{\sigma} = \{\mathbf{u} \in \mathbb{R}^d \mid \mathbf{u} \cdot \mathbf{v} \geq 0, \forall \mathbf{v} \in \sigma\}.$$

Example 8.8 (Dual cones for P_1)**.** On the right we have depicted the cones that we get when we translate each vertex of P_1 to the origin. Their duals are on the left. Notice that the dual cones fit together nicely. Note that we denote the cones on the left by σ_i and the cones on the right by $\hat{\sigma_i}$, which might seem odd from our point of view so far. However, in the literature one typically starts with the picture on the left and then dualizes to produce the picture on the right. Since the dual of the dual of a cone is the original cone, the pictures convey equivalent data, and we follow the standard conventions to ease the transition to other readings.



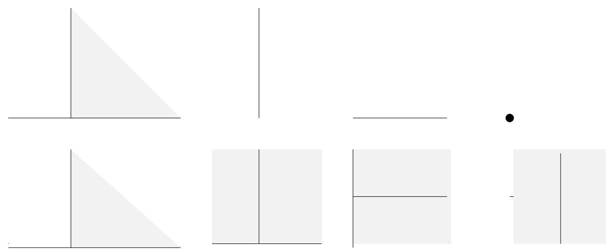
Observe that the rays of the dual cones are exactly the normal vectors to the edges of the polygon. The definitions below give us precise language for describing how these cones fit together.

Definition 8.9. If $\sigma \subset \mathbb{R}^d$ is a cone, we define a *face* of σ to be a set of the form

$$\{\mathbf{v} \in \sigma \mid \mathbf{u} \cdot \mathbf{v} = 0\}$$

for some $\mathbf{u} \in \hat{\sigma}$.

Example 8.10 (Faces of some cone)**.** We have depicted the cone generated by $(1, 0), (0, 1)$ together with each of its faces. Below each face we give its dual.



Definition 8.11. A *fan* $\Delta \subset \mathbb{R}^d$ is a collection of cones satisfying

(1) If $\sigma \in \Delta$ and τ is a face of σ, then $\tau \in \Delta$.
(2) If $\sigma, \tau \in \Delta$, then $\sigma \cap \tau$ is simultanenously a face of σ and a face of τ.

Definition 8.12. If $P \subset \mathbb{R}^d$ is a polytope, its *inner normal fan* is the fan whose maximal cones are the duals of the cones at each vertex of P.

If $P \subset \mathbb{R}^2$ is a lattice polygon, then we can get its inner normal fan as follows. For each edge e of P, translate the normal vector pointing into P to the origin and let ρ_e be the ray spanned by this vector. These rays are the rays of the inner normal fan of P.

8.1. **Exercises.**

(1) Draw the inner normal fans for each of the polygons that we have seen so far.
(2) Pick a fan above. Draw the dual of each cone in the fan.
(3) Blowup warmup: Let $B \subset \mathbb{C}^2 \times \mathbb{P}^1$ given by the equation $x_0 y_1 = x_1 y_0$ be the blowup of $\mathbb{C}^2$ at $\mathbf{0}$ with projection map $\pi : B \to \mathbb{C}^2$. When we blowup we replace $\mathbf{0}$ with a copy of $\mathbb{P}^1$ which we call the *exceptional divisor*. Moreover, the lines through $\mathbf{0}$ get separated in the blowup and instead of intersecting each other at $\mathbf{0}$, each now intersects the new copy of $\mathbb{P}^1$, at a point corresponding to its slope. We can see this explicitly as follows:
    (a) Let $x_1 - mx_0 = 0$ be the equation of a line with nonzero slope. Recall that U_0 is the coordinate patch where $y_0 = 1$. Use the equation of the blowup to rewrite the equation of the line in the coordinates for U_0.
    (b) The new equation should factor in U_0. What does this tell us geometrically about the zero set of this equation in U_0?
    (c) Use the coordinates of U_0 to write down the points of π^{-1} of the line in $\mathbb{C}^2$ defined by $x_1 - mx_0 = 0$. You should see that the inverse image "remembers" the slope.

(d) The closure of the inverse image of the nonzero points on your line is its *strict transform*. Find the coordinates of the intersection of the strict transform of your line with the exceptional divisor.

9. Next

Suppose that $P \subset \mathbb{R}^d$ is a lattice polytope with vertex $\mathbf{a}_0 = \mathbf{0}$. Let σ be the dual of the cone generated by P. The lattice points in the cone generated by the lattice points in P are the exponent vectors of Laurent monomials in $A_\sigma \cong \mathbb{C}[\mathbf{t}^{\mathbf{a}_1}, \dots, \mathbf{t}^{\mathbf{a}_n}] \cong \mathbb{C}[x_1, \dots, x_n]/I_{\mathcal{A}}$.

Remark 9.1. Since we are working in the dimension two today, and we are not going to be looking at the equations of our toric varieties, we will let our semigroup algebras be subsets of $S[x, y, x^{-1}, y^{-1}]$.

Observation 9.2. The ring A_σ corresponds to an "abstract" affine toric variety U_σ. (See section 1.3 of [1] for a treatment of toric varieties which begins with this point of view.) In the setup above, $X_{\mathcal{A}} \subset \mathbb{C}^n$ is an embedding of U_σ into affine space.

Goals:

(1) Understand how a fan Δ tells us how varieties U_σ fit together to give an abstract toric variety X_Δ.
(2) Understand how subdivisions of cones give maps of toric varieties.
(3) Use (1) and (2) to resolve the singularities of a toric surface!

9.1. **Fans and duals II.**

In this section we will see that the dual picture is useful not just because the cones fit together nicely. The point is that faces of the dual cones correspond algebraically to the coordinate rings of open subsets in a nice way. We follow the notation of [1] as closely as possible to help students who wish to transtion from these notes to the other text.

Assumption 9.3. In what follows today we will assume that $d = 2$ throughout. This is the situation that we will depict pictorially. There are natural generalizations for arbitrary d, with the added complication that if $d \geq 3$ specifying a collection of rays does not automatically specify a complete fan.

Since we will be moving between cones and their duals frequently, it will be useful to introduce notation that will help us to remember that a cone and its dual ought to live in different spaces.

Definition 9.4. As in [1], let $M_{\mathbb{R}}$ denote the copy of $\mathbb{R}^2$ that contains P (and the cones at the vertices of P). We let $N_{\mathbb{R}}$ denote the copy of $\mathbb{R}^2$ containing the inner normal fan of P. We denote the lattice in $M_{\mathbb{R}}$ by M and write N for the lattice in $N_{\mathbb{R}}$.

Although A_σ can be realized as $\mathbb{C}[x_1, \dots, x_n]/I_{\mathcal{A}}$ (in many different ways, in fact), it is a ring in its own right independent of its realization as a quotient of a polynomial ring. In fact, A_σ corresponds to an "abstract" affine toric variety $U_\sigma \cong X_{\mathcal{A}}$ via its *spectrum* of prime ideals. For our purposes, we will think only of the set of all maximal ideals of A_σ, for these ideals correspond to points of U_σ in the usual sense (closed points in the language of algebraic geometry). We will denote this abstract affine toric variety by $Specm A_\sigma$, following pg. 14 of [1].

Our first goal today is to understand how to think about these abstract affine toric varieties U_σ and how a fan tells us how they glue together to give an arbitrary toric variety.

Here are some useful facts.

Lemma 9.5. (1) *If $\tau \subset \sigma \subset N$, then $\hat{\sigma} \subset \hat{\tau}$ and $A_\sigma \hookrightarrow A_\tau$.*
(2) *If τ is a face of σ, then A_τ is gotten from A_σ by adjoining the inverse of a monomial in A_σ.*
(3) *The maximal ideals of A_τ correspond precisely to the maximal ideals of A_σ which do not contain the monomial that is inverted.*

Proof. We leave part (1) as an exercise. for part (2), see Proposition 2 on pg. 13 of [1]. Part (3) requires a bit of commutative algebra. Let $f \in A_\sigma$ be the monomial that we invert to get A_τ. If $m \subset A_\tau$ is maximal, let $\overline{m}$ be its intersection with A_σ. If $\overline{m}$ is not maximal, then it is contained in some maximal ideal n. If n does not contain f, then its expansion to A_τ is a nontrivial ideal strictly containing m, which contradicts the maximality of m. Therefore, every maximal ideal containing $\overline{m}$ must contain f. But, A_σ is a Jacobson ring, which means that any prime ideal is the intersection of all of the maximal ideals containing it. As $\overline{m}$ must be prime, this would imply $f \in \overline{m}$, which is a contradiction because then $\overline{m}$ would contain 1. □

Example 9.6 (Cone for $\mathbb{C}^2$)**.** Let $\sigma = \text{cone}\{e_1, e_2\} \subset N_{\mathbb{R}}$ and note that it is its own dual. The vectors in $\hat{\sigma} \subset M_{\mathbb{R}}$ correspond to the monomials in $\mathbb{C}[x, y]$. This is the coordinate ring of $\mathbb{C}^2$.

What do the faces of σ correspond to? The face $\text{cone}\{e_1\}$ has dual equal to the right half-plane. The corresponding algebra is $\mathbb{C}[x, y, \frac{1}{y}]$. This is the ring of polynomial functions on $\mathbb{C}^2 \backslash \{(x, 0) \mid x \in \mathbb{C}\}$. Similarly, the face $\text{cone}\{e_2\}$ corresponds to $\mathbb{C}[x, y, \frac{1}{x}]$ which is the ring of polynomial functions on $\mathbb{C}^2$ minus the y-axis.

9.2. **Limit points and the fan.** Suppose that P is a lattice polygon. How do we find the limit points corresponding to the faces of P?

Let Δ be the inner normal fan of P. Each face of P corresponds to a cone in Δ in a natural way. The maximal face of P corresponds to the origin. If e is an edge of P, it corresponds to the 1-dimensional cone spanned by an inner normal vector to e. If v is a vertex of P it corresponds to the 2-dimensional cone that is dual to the 2-dimensional cone generated by P when we translate v to the origin.

Theorem 9.7 (Claim 1 on pg. 38 of [1])**.** *Let $P \subset \mathbb{R}^2$ be a lattice polygon with inner normal fan Δ. For a face Q of P, let $\sigma_Q \in \Delta$ be the cone corresponding Q. If $\mathbf{v}$ is in the relative interior of σ_Q, then*

$$\lim_{z \to 0} \psi_P(\lambda_{\mathbf{v}}(z))$$

is the distinguished orbit representative of the orbit corresponding to Q.

Proof. See [1]. □

9.3. **Fans and gluing varieties.** Suppose that we have a fan Δ in N. For each cone $\sigma \in \Delta$ we have a ring A_σ that is the algebra on monomials with exponent vectors in $\hat{\sigma}$.

If cones σ_1 and σ_2 intersect in a face τ, then $A_{\sigma_i} \hookrightarrow A_\tau$ for each i. These ring maps correspond to $U_\tau \subset U_{\sigma_i}$ as an open subset. So, we see that the cones in N correspond to affine varieties which glue together over open sets corresponding to faces of cones in N.

Example 9.8 ($\mathbb{P}^2$)**.** The three cones $\sigma_0, \sigma_1, \sigma_2$ correspond to affine varieties U_{σ_i} which correspond to the standard affine open cover of $\mathbb{P}^2$ by coordinate patches U_i. Let's see this and see how the ring maps give rise to the usual patching maps.

We see that $A_{\sigma_0} = \mathbb{C}[x, y], A_{\sigma_1} = \mathbb{C}[\frac{1}{x}, \frac{y}{x}]$, and $A_{\sigma_2} = \mathbb{C}[\frac{x}{y}, \frac{1}{y}]$. Each of these rings is the affine coordinate ring of a copy of $\mathbb{C}^2$ as they each require two generators that do not satisfy any relations.

The intersection of σ_0 and σ_1 is the face τ equal to the nonnegative y-axis. Therefore, the dual of τ is the upper halfplane and $A_\tau = \mathbb{C}[x, \frac{1}{x}, y]$. The maximal ideals of A_τ have the form $\langle x - a, y - b \rangle$. Notice however, that such an ideal contains $1 - \frac{a}{x}$. Therefore, if $\langle x-a, y-b \rangle$ is a maximal ideal, $a \neq 0$. (Otherwise, it contains 1 so is equal to the whole ring.) i.e., $U_\tau \cong \mathbb{C}^* \times \mathbb{C}$.

We know that $A_{\sigma_0} \hookrightarrow A_\tau$. In the map of varieties from $U_\tau \to U_{\sigma_0}$ the point $(a, b) \in U_\tau$ goes to the point $(a, b) \in U_{\sigma_0}$ since the inverse image of $\langle x - a, y - b \rangle \subset A_\tau$ is just the ideal $\langle x - a, y - b \rangle \subset A_{\sigma_0}$

We also have $A_{\sigma_1} \hookrightarrow A_\tau$. To see how to pull back the ideal $\langle x-a, y-b\rangle$ we do a little computation.

$$\langle x-a, y-b\rangle = \langle 1-\frac{a}{x}, \frac{y}{x} - \frac{b}{x}\rangle = \langle \frac{1}{a} - \frac{1}{x}, \frac{y}{x} - \frac{b}{a}\rangle$$

This shows us that $(a,b) \in U_\tau$ is the image of $(\frac{1}{a}, \frac{b}{a}) \in U_{\sigma_1}$. (Or, that $(c,d) \in U_{\sigma_1}$ maps to $(\frac{1}{c}, \frac{d}{c}) \in U_\tau$.) We see that the points (a,b) in U_{σ_0} with $a \neq 0$ glue to the points $(\frac{1}{a}, \frac{b}{a}) \in U_{\sigma_1}$.

We'll check the other transitions in the exercises.

9.4. **Refining a fan.**

Example 9.9 ($\mathbb{P}^2$ vs X_{P_3})**.** Consider the inner normal fans of P_1 and P_3. Notice that the only difference is that one of the cones has been split in two.

Definition 9.10. We say that a fan Δ' is a *refinement* of Δ if every cone in Δ is the union of cones in Δ'.

The important point is that if $\sigma' \subset \sigma$, then $A_\sigma \hookrightarrow A_{\sigma'}$. Therefore, we get a map of varieties going the other direction. More precisely,

Proposition 9.11. *If Δ' is a refinement of Δ, then there is birational morphism $X_{\Delta'} \to X_\Delta$.*

This is discussed in [1]. See the exercise and hint on pg. 18 and more on pgs. 22-23.

Example 9.12 (The blowup of $\mathbb{C}^2$ at $\mathbf{0}$)**.** Consider the fan Δ consisting of cone$\{(1,0),(0,1)\}$ and all of its faces. Let Δ' be the fan gotten by subdividing the maximal cone of Δ by the nonnegative span of $(1,1)$. So, Δ' is a refinement of Δ.

Let σ be the maximal cone of Δ so that $A_\sigma = \mathbb{C}[x,y]$. Let $\sigma_0 =$ cone$\{(0,1),(1,1)\}$ and $\sigma_1 =$ cone$\{(1,1),(1,0)\}$. We see that $A_{\sigma_0} = \mathbb{C}[x, \frac{y}{x}]$ and $A_{\sigma_1} = \mathbb{C}[\frac{x}{y}, y]$. Each of these rings is the coordinate ring of a copy of $\mathbb{C}^2$. Moreover, since $A_\sigma \hookrightarrow A_{\sigma_i}$ we get maps of affine varieties $U_{\sigma_i} \to U_\sigma$.

Let's see what these maps do. The point $(a,b) \in U_{\sigma_0}$ corresponds to maximal ideal $\langle x-a, \frac{y}{x} - b\rangle$. Since

$$\langle x-a, \frac{y}{x} - b\rangle = \langle x-a, y-bx\rangle = \langle x-a, y-ba\rangle$$

we see that $(a,b) \mapsto (a, ab)$ and that the points $(0,b)$ all map to the origin.

Similarly, a point $(c,d) \in U_{\sigma_1}$ corresponds to the ideal $\langle \frac{x}{y} - a, y-b\rangle$ which pulls back to the ideal $\langle x-ab, y-b\rangle$ in A_σ. Therefore, $(c,d) \mapsto (cd, d)$ and all points of the form $(c,0)$ go to the origin.

In fact, the U_{σ_i} are exactly the standard affine patches covering the blowup of $\mathbb{C}^2$ at the origin. To see this, let's glue them together. The cones intersect in the ray τ spanned by $(1,1)$ so $A_\tau = \mathbb{C}[x, \frac{x}{y}, \frac{y}{x}]$.

Let's see which points $(a,b) \in U_{\sigma_0}$ gets glued to points of U_{σ_1}. The ideal $\langle x-a, \frac{y}{x}-b\rangle \subset A_{\sigma_0}$ is the inverse image of an ideal with the same generators in A_τ. The ideal in A_τ can be rewritten so that we can tell what its inverse image in A_{σ_1} is as follows. We want to see an ideal of the form $\langle \frac{x}{y} - c, y-d\rangle \subset A_{\sigma_1}$.

Since

$$-\frac{1}{b}\frac{x}{y}(\frac{y}{x} - b) = \frac{x}{y} - \frac{1}{b}$$

and

$$\frac{y}{x}(x-a) + a(\frac{y}{x} - b) = y - ab$$

we see that $(a,b) \in U_{\sigma_0}$ corresponds to $(\frac{1}{b}, ab) \in U_{\sigma_1}$ and that for this correspondence to hold, $b \neq 0$.

Here is a chart that summarizes the information of which points from U_{σ_0} and U_{σ_1} get glued together and where they go in U_σ under the "blow down" map.

U_{σ_0}	U_{σ_1}	U_σ	conditions for gluing
(a,b)	$(\frac{1}{b}, ab)$	(a, ab)	$b \neq 0$
$(a,0)$	$-$	$(a,0)$	$-$
$(0,b)$	$(\frac{1}{b}, 0)$	$(0,0)$	$b \neq 0$
$(cd, \frac{1}{d})$	(c,d)	(cd,d)	$d \neq 0$
$-$	$(c,0)$	$(0,0)$	$-$
$(0, \frac{1}{d})$	$(0,d)$	$(0,d)$	$d \neq 0$

10. Smoothness I

Fact 10.1. If P is a lattice polygon, then the singular points of X_P are isolated points. This implies that if p is a singular point its orbit must be zero-dimensional and hence that p is a fixed point.

Suppose that P is a lattice polygon and v is a vertex which we may assume is at the origin in $\mathbb{R}^2$.

Proposition 10.2. *If we can find an element of $GL_2(\mathbb{Z})$ that brings the edges incident at v to the positive coordinate axes of $\mathbb{R}^2$, then X_P is smooth at the fixed point corresponding to v.*

Proof. As P lies in the first quadrant of $\mathbb{R}^2$, the fixed point corresponding to v is the image of $(0,0)$ in the natural extension of ψ_P to $\mathbb{C}^2$. Moreover, t_1 and t_2 appear as coordinates of ψ_P. Therefore, $\frac{\partial \psi_P}{\partial t_1}|_{(0,0)}$

and $\frac{\partial \psi_P}{\partial t_2}|_{(0,0)}$ are linearly independent. These two vector span a 2-plane that is tangent to X_P at $\psi_P(0,0)$. Thus, the point is smooth. □

In the next lecture we will work on developing the local vocabulary to discuss smoothness algebraically and we will eventually see that the proposition above is an if and only if statement.

11. Smoothness II

Suppose we want to investigate whether the point at the origin in an affine variety $X_{\mathcal{A}}$ is singular. In $\mathbb{C}[x_1, \ldots, x_n]/I_{\mathcal{A}}$, the ideal corresponding to the origin is $m = \langle x_1, \ldots, x_n \rangle$. (Of course, although $x_1, \ldots, x_n$ form a set of minimal generators for the ideal of the origin in $\mathbb{C}^n$, they may not be minimial generators of the ideal they generate in the quotient ring.)

In the theorem below we translate the computation of $\dim_{\mathbb{C}} m/m^2$ into terminology using only the lattice.

Theorem 11.1 (pgs. 28-29 in [1])**.** *Let A be a $d \times n$ integer matrix and σ be the cone generated by the columns of A. Assume that the columns of A generate the set $\sigma \cap \mathbb{Z}^d$ as a semigroup, i.e., that every element of $\sigma \cap \mathbb{Z}^d$ is a finite sum of some subset of the columns of A.*

The point $\mathbf{0} \in X_{\mathcal{A}}$ is smooth if and only if the dimension of $X_{\mathcal{A}}$ is equal to the size of the set

$$\{\mathbf{v} \in (\sigma \cap \mathbb{Z}^d) \mid \mathbf{v} \neq 0,\ \mathbf{v} \neq \mathbf{u} + \mathbf{w},\ \mathbf{u}, \mathbf{w} \in (\sigma \cap \mathbb{Z}^d) \backslash \{\mathbf{0}\}\}.$$

Proof. Let $m_{\mathbf{0}}$ denote the ideal of the origin in $\mathbb{C}[\mathbf{t}^{\mathbf{a}_1}, \ldots, \mathbf{t}^{\mathbf{a}_n}] \cong \mathbb{C}[x_1, \ldots, x_n]/I_{\mathcal{A}}$. Note that $m_{\mathbf{0}}$ is spanned by monomials with exponents in $(\sigma \cap \mathbb{Z}^d) \backslash \{\mathbf{0}\}$. The ideal $m_{\mathbf{0}}^2$ is spanned by products of monomials in $m_{\mathbf{0}}$. Hence, the exponent vectors of elements of $m_{\mathbf{0}}^2$ are precisely the set of elements in in $(\sigma \cap \mathbb{Z}^d) \backslash \{\mathbf{0}\}$ that are the nontrivial sum of two other such elements.

Therefore, to compute the vector space dimension of $m_{\mathbf{0}}/m_{\mathbf{0}}^2$, we just need to count the vectors in $(\sigma \cap \mathbb{Z}^d) \backslash \{\mathbf{0}\}$ that cannot be written as the nontrivial sum of two other elements in $(\sigma \cap \mathbb{Z}^d) \backslash \{\mathbf{0}\}$. By definition the variety $X_{\mathcal{A}}$ is smooth at $\mathbf{0}$ if and only if $\dim X_{\mathcal{A}} = \dim_{\mathbb{C}} m_{\mathbf{0}}/m_{\mathbf{0}}^2$, so we are done. □

Fact 11.2. If $\sigma \subset \mathbb{R}^d$ is a d-dimensional cone, then $\sigma \cap \mathbb{Z}^d$ generates $\mathbb{Z}^d$ as a group. Essentially this is because there are no "holes" in the cone σ. For a converse to this see the Exercise on pg. 19 of [1].

Corollary 11.3 (Proposition on pg. 29 of [1])**.** *The variety $X_{\mathcal{A}}$ is smooth at $\mathbf{0}$ if and only if the first lattice vectors along the rays of σ are a $\mathbb{Z}$-basis for $\mathbb{Z}^d$.*

Proof. The vector space $m_{\mathbf{0}}/m_{\mathbf{0}}^2$ has a basis consisting of monomials, and this basis must contain the first lattice vectors along the rays of σ. If σ is d-dimensional, it must have at least d generators. This vector space has dimension $d = \dim X_{\mathcal{A}}$ if and only if $\mathbf{0}$ is a smooth point. Therefore, $\mathbf{0}$ is a smooth point if and only if $m_{\mathbf{0}}/m_{\mathbf{0}}^2$ has a basis consisting of the first lattice vectors along the rays of σ. These vectors clearly generate $\sigma \cap \mathbb{Z}^d$ if all of the elements of $m_{\mathbf{0}}^2$ are the sum of at least two of them. Since $\sigma \cap \mathbb{Z}^d$ generates $\mathbb{Z}^d$ as a group, these lattice vectors must also generate $\mathbb{Z}^d$ as a group. □

Here is an example of a singular point on a toric surface.

Example 11.4 (Cone over a conic)**.** Let P be the convex hull of $\{(0,0),(1,0),(1,2)\}$ depicted below.

We see the region containing the monomials in $m_{\mathbf{0}}$ and the region containing the monomials in $m_{\mathbf{0}}^2$. The three black dots in the cone on the right represent the monomials in $m_{\mathbf{0}}$ that are not in $m_{\mathbf{0}}^2$. We see that $\dim_{\mathbb{C}} m_{\mathbf{0}}/m_{\mathbf{0}}^2 = 3 \neq \dim X_P = 2$. Therefore we see that the fixed point corresponding to $\mathbf{0}$ is a singular point.

12. Toric resolution of singularities

We have seen that a toric surface is nonsingular at a fixed point corresponding to vertex v if and only if the first lattice vectors along the cone edges meetig at v formed a $\mathbb{Z}$-basis for $\mathbb{Z}^2$.

Question 12.1. Can we resolve these singularities staying in the toric world? i.e., can we find a smooth toric surface that maps birationally to a given singular toric variety?

The answer is – yes! Given any fan, we can subdivide cones until each cone corresponds to a smooth affine toric patch. We get a toric resolution of singularities. We'll work out examples of toric resolution of singularities of surfaces tomorrow.

Remark 12.2. Recall our basic setup. We have a lattice polytope $P \subset M_{\mathbb{R}} \cong \mathbb{R}^d$ with $P \cap M = \{\mathbf{a}_0 = \mathbf{0}, \dots, \mathbf{a}_n\}$. Then if $A = (\mathbf{a}_1 \cdots \mathbf{a}_n)$, the affine variety $X_{\mathcal{A}} = X_P \cap U_0$ has coordinate ring $\mathbb{C}[\mathbf{t}^{\mathbf{a}_1}, \dots, \mathbf{t}^{\mathbf{a}_n}] \cong \mathbb{C}[x_1, \dots, x_n]$.

Note: Let σ be the dual of the cone generated by P. For the ring $\mathbb{C}[\mathbf{t}^{\mathbf{a}_1}, \dots, \mathbf{t}^{\mathbf{a}_n}]$ to be the algebra spanned by monomials whose exponent vectors are the elements of $\hat{\sigma} \cap M$, we must be able to write every element of $\hat{\sigma} \cap M$ as a sum of elements from the set $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$. This is the same as saying that the lattice points in P must generate $\hat{\sigma} \cap M$ as a semigroup.

We need the condition above to hold for P at every vertex in order to say that X_P is isomorphic to the abstract toric variety that we associate to inner normal fan Δ of P. In more advanced language, we need P to correspond to a very ample divisor. This is always true if P is a polygon or if P has an inner normal fan corresponding to a smooth variety. However, in dimensions greater than 2, P may correspond to a divisor that is merely ample on the abstract variety associated to its inner normal fan. (See pgs. 70-72 of [1] for a discussion.)

The upshot of this remark is that everything we've done works in the 2-dimensional case, and in higher dimensions more care is needed.

In today's lecture we are following sections 2.2, 2.5, an 2.6 of [1].

Lemma 12.3. *Let σ be a 2-dimensional cone and $\mathbf{v}_1, \mathbf{v}_2$ be the first lattice vectors along its 1-dimensional faces (listed counter clockwise). Then there exists an invertible 2×2 integer matrix B so that $B\mathbf{v}_1 = (m, -k)$ where $0 \le k < m$ are integers and $B\mathbf{v}_2 = (0, l)$.*

Proof. This is an exercise. □

Example 12.4. Consider the cone below where $\mathbf{v}_1 = (2, -1)$ and $\mathbf{v}_2 = (0, 1)$. Since $\det \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} = 2$, the vectors are not a $\mathbb{Z}$-basis for $\mathbb{Z}^2$. Therefore, the origin is a singular point of the corresponding affine variety.

Subdivide this cone by adding in the ray spanned by $(1, 0)$. Taking since $\det \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} = 1$ and $\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$, the affine varieties associated to the new 2-dimensional cones are both smooth!

From yesterday, we know that adding in this new ray exactly corresponds to blowing up the origin in our original affine variety. In this case, we get a smooth variety after just one blowup.

In some cases, we may need to repeat this subdivision procedure multiple times, but we can give an algorithm (a set of steps which terminates) to resolve any singularity .

Algorithm 12.5. INPUT: A cone σ generated by $(m_0, -k_0)$, with $0 \leq -k_0 < m_0$ and $(0, 1)$.

OUTPUT: A fan Δ corresponding to a smooth toric variety X_Δ that is gotten by starting with U_σ and then blowing up finitely many times.

Let Δ be the fan consising of σ and all of its faces.

(1) Set Δ equal to the subdivision of the current fan Δ gotten by adding in the ray spanned by $(1, 0)$.
(2) Let $\Delta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Delta$. (Rotate Δ by 90 degrees.)
(3) Let $\Delta = \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix} \Delta$ for some value of a that takes (k_i, m_i) to $(m_{i+1}, -k_{i+1})$ where $m_{i+1} = k_i$ and $0 \leq k_{i+1} < m_{i+1}$. (This *shears* the plane.)
(4) If $k_{i+1} = 0$, then stop. Otherwise, let $i = i + 1$ and repeat.

Example 12.6. Let σ be the cone generated by $(3, -2)$ and $(0, 1)$. After we add the ray generated by $(1, 0)$, the cone in the first quadrant is smooth, but the one below the first quadrant is not. After the rotation in step (2) of the algorithm, we shear by $\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$.

The nonsmooth cone is now generated by $(2, -1)$ and $(0, 1)$. Applying the algorithm again, we insert the ray spanned by $(1, 0)$. We now check that each cone is generated by a $\mathbb{Z}$-basis for $\mathbb{Z}^2$.

Theorem 12.7. *If X_Δ is a smooth projective toric surface, then it can be constructed by blowing up finitely many times starting with either $\mathbb{P}^2$ or a* Hirzebruch surface *corresponding to the fan with rays* $(1, 0), (0, 1), (-1, t), (0, -1)$.

Proof. A proof is outlined in the exercises in section 2.5 of [1]. □

References

[1] William Fulton, *Introduction to Toric Varieties*, Annals of Math. studies no. 131, Princeton Univ. Press, Princeton, 1993.
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