TORIC SURFACES
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Introduction

These notes are meant to supplement the lectures in a one-week course introducting toric varieties at an undergraduate level. We take the general philosophy of [2], first introducing varieties embedded in affine and projective space, and arriving at the abstract definition of a toric variety associated to a fan only after exploring many concrete examples. In the first lectures we follow the notation of [2] and [3], which the reader who would like more detail than we have presented here might wish to consult. As we move towards greater abstraction, we adopt the language of [1].

1. Day 1

Our goals for the day are to understand how to

(1) compute the ideal of an affine toric variety
(2) construct a projective toric variety from a lattice polygon
(3) determine if two lattice polygons correspond to the toric surface in $\mathbb{P}^n$
(4) compute the ideal of a projective toric variety

1.1. **Affine toric varieties.** Throughout sections 1.1.1 and 1.1.2, we follow Sturmfels's book [3].

1.1.1. *From lattice points to monomial mappings: defining X_A.*

Definition 1.1. We call $\mathbb{Z}^d \subset \mathbb{R}^d$ the d-dimensional *lattice* and call elements of $\mathbb{Z}^d$ *lattice points.*

Suppose that we have indeterminates $x_1, \ldots, x_d$. If we have a lattice point $\mathbf{a} = (a_1, \ldots a_d)$ then we can associate to it a *Laurent monomial* in the $x_i's$: $x_1^{a_1} \cdots x_n^{a_d}$. (We call this a *Laurent* monomial because we allow negative exponents.) We often write $\mathbf{x}^{\mathbf{a}}$ using vector notation instead of writing out each indeterminate and each exponent. Suppose

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we have a finite ordered set $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{Z}^d$. We'll assume that $\mathbf{a}_i = (a_{i,1}, \ldots, a_{i,d})$. Let A be the $d \times n$ matrix $A = (\mathbf{a}_1 \ \cdots \ \mathbf{a}_n)$.

If we are allowing monomials with negative exponents, none of the elements in the domain can have a zero coordinate. Therefore, A gives a map $\phi_A : (\mathbb{C}^*)^d \to \mathbb{C}^n$ given by $\mathbf{t} = (t_1, \ldots, t_d) \mapsto (\mathbf{t}^{\mathbf{a}_1}, \ldots, \mathbf{t}^{\mathbf{a}_n})$.

Definition 1.2. The *affine toric variety* associated to A, denoted X_A, is the Zariski closure of the image of ϕ_A. The monomials associated to columns of A parameterize a dense open subset of X_A.

Example 1.3. Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

This corresponds to the map $\phi_A : (\mathbb{C}^*)^2 \to \mathbb{C}^3$ given by $\phi_A(t_1, t_2) = (t_1, t_2, t_1 t_2)$.

1.1.2. *Computing the ideal of X_A.* A variety is not just a set of points in projective space. An algebraic variety is a set that arises as the set of all points that are solutions to some finite set of polynomial equations. In this section we will learn how to compute the ideal of the variety X_A.

To do this, let's think carefully about what it means for a polynomial on the ambient space to vanish at a point $\phi_A(\mathbf{t})$. If $f(x_1, \ldots, x_n)$ is a polynomial, then we need to compute $f(\phi_A(\mathbf{t}))$. This is done monomial-by-monomial. So, let's focus on what it really means to substitute $\phi_A(\mathbf{t})$ into a monomial $\mathbf{x}^{\mathbf{u}}$. This is best done in an example. We will see that if $f = \mathbf{x}^{\mathbf{u}}$, the $f(\phi_A(\mathbf{t})) = \mathbf{t}^{A\mathbf{u}}$.

Let $f(x_1, x_2, x_3) = x_3 - x_1 x_2 = \mathbf{x}^{(0,0,1)} - \mathbf{x}^{(1,1,0)}$. Then

$$\begin{aligned} f(\phi_A(\mathbf{t})) =& f(\mathbf{t}^{(1,0)}, \mathbf{t}^{(0,1)}, \mathbf{t}^{(1,1)}) \\ =& \mathbf{t}^{0\cdot(1,0)}\mathbf{t}^{0\cdot(0,1)}\mathbf{t}^{1\cdot(1,1)} - \mathbf{t}^{1\cdot(1,0)}\mathbf{t}^{1\cdot(0,1)}\mathbf{t}^{0\cdot(1,1)} \\ =& \mathbf{t}^{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} - \mathbf{t}^{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}} \end{aligned}$$

What we observe from above is that substituting a monomial map into a monomial $\mathbf{x}^{\mathbf{u}}$ is equivalent to applying a linear transformation to $\mathbf{u}$ using the matrix of the monomial map. From this it is clear that if we have a binomial $f(\mathbf{x}) = \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$, then $f(\phi_A(\mathbf{t}))$ is identically zero exactly when $A\mathbf{u} = A\mathbf{v}$. In fact, we have the following theorem:

Theorem 1.4 (Lemma 4.1 in [3])**.** *Suppose A is an $d \times n$ integer matrix. Let I_{X_A} be the ideal of the set X_A. Then I_{X_A} is equal to the ideal*

$$I_A := \langle \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \mid \mathbf{u} - \mathbf{v} \in \ker A,\ \mathbf{u}, \mathbf{v} \in \mathbb{Z}^n_{\geq 0} \rangle.$$

Proof. We will show that I_A is contained in I_{X_A} with a computation. If $f \in I_A$, then it is a linear combination (with polynomial coefficients) of elements of the form $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ where $A\mathbf{u} = A\mathbf{v}$. For each of these binomials

$$\mathbf{t}^{A\mathbf{u}} - \mathbf{t}^{A\mathbf{v}} = 0$$

becuase $A\mathbf{u} = A\mathbf{v}$. Therefore, $f(\phi_A(\mathbf{t})) = 0$ and $f \in I_{X_A}$.

Next, we will show that every element of I_{X_A} is in I_A. Let U be the set of all polynomials in I_{X_A} that are not in I_A. For each $f \in U$, consider the lexicographic leading term of f, i.e., the term that would come first in the dictionary. If U is not empty, then there is some nonzero f in U whose leading term is smallest. (The set of exponent vectors is bounded below by $\mathbf{0}$ and is discrete.)

Since $f \in I_A$, $f(\phi_A(\mathbf{t})) = 0$. Without loss of generality, assume that the leading term of f is $\mathbf{x}^{\mathbf{u}}$, so that the coefficient of the leading term is 1. If we expand $f(\phi_A(\mathbf{t}))$, we will see a term of the form $\mathbf{t}^{A\mathbf{u}}$. Since this term cancels when we simplify, there must be at least one term of the form $\gamma \mathbf{t}^{A\mathbf{v}}$ with $A\mathbf{u} = A\mathbf{v}$ in the expression. Now define $f' = f - (\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}})$. Notice that the leading term of f' is lexicographically smaller than the leading term of f as we have gotten rid of our original leading term $\mathbf{x}^{\mathbf{u}}$ and as $\mathbf{x}^{\mathbf{v}}$ appeared with nonzero coefficient in f, we have not introduced any additional monomials. If f' is in U, then we have contradicted the minimality of our choice of f. If f' is not in U, then neither is f which is again a contradiction. Therefore, U must be empty and the claim is proved. $\square$

Corollary 1.5 (Corollary 4.3 in [3])**.** *Assume the hypotheses of Theorem 1.4. Suppose that $\mathbf{u} \in \ker A$. Write $\mathbf{u} = \mathbf{u}^+ - \mathbf{u}^-$, where $\mathbf{u}^+, \mathbf{u}_- \in \mathbb{Z}^n_{\geq 0}$. Then $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}_-} \in I_{X_A}$.*

1.2. **Polygons and projective toric varieties.** A finite set of points in $\mathbb{R}^n$ gives rise to a polytope via the following construction.

Definition 1.6. Let $X =\subset \mathbb{R}^n$. The *convex hull* of X is

$$\operatorname{conv} X = \{r_1\mathbf{a}_1 + \cdots + r_m\mathbf{a}_m \mid \mathbf{a}_i \in X,\ 0 \leq r_i,\ \sum r_i = 1\}.$$

If X is a finite set of points, $\operatorname{conv} X$ is a convex *polytope.* If $X \subset \mathbb{Z}^n \subset \mathbb{R}^n$ then $\operatorname{conv} X$ is a *lattice polytope.* If $n = 2$, and the points in X are not all collinear, then X is a lattice polygon.

We shall see below that it is easy to explain how a lattice polygon defines a parameterization of a dense open subset of a toric variety. We take the point of view of [2].

Given a lattice polygon P, fix an ordering on $P \cap \mathbb{Z}^2$ so that we may list the points $\{\mathbf{m}_0, \ldots, \mathbf{m}_n\}$. Let $\mathbb{C}^*$ denote $\mathbb{C}\backslash\{0\}$, and define $\phi_P : (\mathbb{C}^*)^2 \to \mathbb{P}^n$ by $\phi_P(\mathbf{t}) = [\mathbf{t}^{\mathbf{m}_0} : \cdots : \mathbf{t}^{\mathbf{m}_n}]$. Note that since we do not allow any $t_i = 0$, this map is defined at every point of $(\mathbb{C}^*)^2$.

Definition 1.7. The projective toric variety X_P associated to a lattice polytope P is the closure of the image of ϕ_P in $\mathbb{P}^n$. The image of a permutation of the coordinates of ϕ_P is isomorphic to X_P via a linear change of coordinates on $\mathbb{P}^n$. (See pg. 66 of [1], pg. 166 of [2], and pgs. 31 and 36 in [3].)

1.2.1. *The ideal of X_P.* Think carefully now. When we write $\phi_P(\mathbf{t})$ we are using a specific choice of coordinates. If a homogeneous polynomial is supposed to vanish at $\phi_P(\mathbf{t})$ viewed as a point in projective space, then it should vanish at $\lambda\phi_P(\mathbf{t})$ for any nonzero λ.

In other words, we should introduce a new indeterminate λ and multiply through in each coordinate. Let $\mathbf{a}_i$ be a 3-dimensional vector with first coordinate 1 and second two coordinates given by $\mathbf{m}_i$. Then the ideal I_A defined in §1.1.2 is the ideal of X_P. If the matrix A comes from a polygon in this way, we denote the ideal by I_P.

1.2.2. *Equivalences of polygons.*

Question 1.8. Can two distinct polytopes P, Q give rise to the same variety in $\mathbb{P}^n$?

Example 1.9. Consider the polygons $P = \operatorname{conv}\{(0,0),(1,0),(0,1),(1,1)\}$ and $Q = \operatorname{conv}\{(-1,0),(0,0),(-1,1),(0,1)\}$. Notice that $Q = TP$ where $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We see that $\phi_P(\mathbf{t}) = [1 : t_1 : t_2 : t_1t_2]$ and

$$\phi_Q(\mathbf{t}) = [1 : t_2 : \frac{1}{t_1} : \frac{t_2}{t_1}] = \phi_P(t_2, \frac{1}{t_1}),$$

and that the map $(t_1, t_2) \mapsto (t_2, \frac{1}{t_1})$ is an isomorphims of $(\mathbb{C}^*)$ with itself (as a group under coordinatewise multiplication).

Definition 1.10. Let P and Q be lattice polygons. Define $P \equiv Q$ if there exists a translation vector w and an integer matrix $B \in GL_2(\mathbb{Z})$ such that $Q = w + BP$.

Theorem 1.11. *Let $P, Q \subset \mathbb{R}^d$ be lattice polygons with $P \equiv Q$. Then $X_P = X_Q$.*

Proof. We will show that if $Q = TP$ for some $T \in GL_2(\mathbb{Z})$, then $X_P = X_Q$. To complete the proof, you will show that if $Q = \mathbf{w} + P$, then $X_P = X_Q$.

Write $T = (\mathbf{T}_1 \cdots \mathbf{T}_d)$. Note that

$$\mathbf{t}^{T\mathbf{m}} = \mathbf{t}^{m_1 \cdot \mathbf{T}_1 + \cdots + m_d \cdot \mathbf{T}_d} = (\mathbf{t}^{\mathbf{T}_1}, \ldots, \mathbf{t}^{\mathbf{T}_d})^{\mathbf{m}}.$$

Hence, if $Q = TP$, then $\phi_Q(\mathbf{t}) = [\mathbf{t}^{T\mathbf{m}_0} : \cdots : \mathbf{t}^{T\mathbf{m}_n}] = \phi_P(\mathbf{t}^{\mathbf{T}_1}, \ldots, \mathbf{t}^{\mathbf{T}_d})$. This shows that the image of ϕ_Q is contained in the image of ϕ_P. Analogously, $P = T^{-1}Q$, so the image of ϕ_P is contained in the image of ϕ_Q. Therefore, the images coincide and $X_P = X_Q$.

$\square$

Remark 1.12. See Proposition 1.2 in part II of [2] for a more general version of this theorem.

1.3. Exercises.

(1) Draw the convex hull of each set below.
   (a) $\{(0,0), (1,0), (2,0), (0,1), (1,1)\}$
   (b) $\{(0,0), (1,0), (0,1)\}$
   (c) $\{(0,0), (1,0), (2,0), (1,1), (0,1), (0,2)\}$
   (d) $\{(0,0), (1,0), (1,2)\}$
   (e) $\{(m,0) \mid m \in \mathbb{Z}_{>0}\} \cup \{(0,m) \mid m \in \mathbb{Z}_{>0}\}$

(2) Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}.$$

   (a) Let $\mathbf{v}_1 = (1, -2, 1, 0), \mathbf{v}_2 = (1, -1, -1, 1), \mathbf{v}_3 = (0, 1, -2, 1)$ Show that these vectors are in $\ker A$.
   (b) Show that $\mathbf{v}_1$ is in the span of $\mathbf{v}_2$ and $\mathbf{v}_3$.
   (c) Use Corollary 1.5 to write down the binomials corresponding to $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.
   (d) Is the binomial associated to $\mathbf{v}_1$ in the ideal generated by the binomials associated to $\mathbf{v}_2$ and $\mathbf{v}_3$? Can you explain your answer and how it relates to part (a)?
   (e) The idea I_A in this exercise defines a very famous variety. Have you seen it before?

(3) Consider the integer matrices

$$A_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 & 2 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}, A_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

If I_{A_i} is nonzero, find at least three elements in I_{A_i} by hand.

(4) Let P denote the polygon in Exercise 1 (a). Label the vertices v_1, v_2, v_3, v_4. For each i, let P_i denote the translate of P with $v_i = \mathbf{0}$, and write down ϕ_{P_i}. (Keep a fixed ordering on the lattice points in P throughout.) Explain how these maps are related.

(5) Using the previous exercise as your guide, prove that if P is a lattice polygon and Q is an integral translate of P, then the image of ϕ_P is equal to the image of ϕ_Q.

(6) Show that if B is an $n \times n$ integer matrix, then it has an inverse with integer entries if and only if $\det B = \pm 1$. We will denote the set of all $n \times n$ invertible integer matrices by $GL_n(\mathbb{Z})$.

References

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