1. Day 2

In today’s lecture you will learn how to
(1) realize several classical examples ($\mathbb{P}^1 \times \mathbb{P}^1$, the Veronese surface, $\mathbb{P}^2$, a rational normal scroll) as toric varieties using polygons.
(2) work with the action of the torus on $X$ and compute the orbits of this action.
(3) visualize the limit points of our examples geometrically.
(4) determine if a toric surface is smooth by computing tangent planes.

1.1. Examples. We give four examples of toric surfaces below.

The varieties can be described via lattice polygons:

\[ P_1 = \quad P_2 = \]
\[ P_3 = \quad P_4 = \]

or via the corresponding integer matrices:

\[ A_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \]
\[ A_3 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \]

The author is partially supported by NSF grant DMS 0600471 and the Clare Boothe Luce Program.
1.2. **The torus action.** In this section we will study the action of the torus \((\mathbb{C}^*)^d\) on a variety \(X_P\) and compute the orbits of the torus action. Restricting to the case of surfaces for simplicity, we will see that this action helps us to decompose the limit points of \(X_P\) into pieces that fit together in a nice way.

Recall the following definitions from group theory.

**Definition 1.1.** A group \(G\) acts on a set \(S\) if we have defined \(g \cdot s \in S\) for every \(g \in G\) and \(s \in S\) so that

1. \(e \cdot s = s\), where \(e \in G\) is the identity element.
2. \((gh) \cdot s = g \cdot (h \cdot s)\).

Given an element \(s \in S\), the orbit of \(s\) is

\[
\{ t \in S \mid \exists g \in G \text{ with } t = g \cdot s \}.
\]

If \(s\) and \(t\) are in the same orbit, we write \(s \equiv t\).

Given a lattice polytope \(P\), we get an action of \(T = (\mathbb{C}^*)^d\) on the image of \(\phi_P\). If \(g \in T\), define \(g \cdot t^m_i\) by \(g^{m_i} \cdot t^m_i\). Note that \(g \cdot \phi_P(t) = \phi_P(g t_1, \ldots, g t_d)\). This shows that \(T\) leaves the image of \(\phi_P\) fixed. Clearly, this action also extends to projective space: \(g \cdot [x_0 : \cdots : x_n] = [g^{m_0} x_0 : \cdots : g^{m_n} x_n]\).

Let’s look at examples 1 and 4.

**Example 1.2** (The triangle \(P_1\)). The set \(P_1 \cap \mathbb{Z}^2 = \{(0, 0), (1, 0), (0, 1)\}\). These lattice vectors give the map \(\phi_{P_1} : (\mathbb{C}^*)^2 \rightarrow \mathbb{P}^2\) given by \(t \mapsto [1 : t_1 : t_2]\). The torus action is \(g \cdot [1 : t_1 : t_2] = [1 : g t_1 : g t_2]\).

First we’ll check that the action is transitive on the image of \(\phi_{P_1}\). Indeed, suppose that \([1 : t_1 : t_2]\) and \([1 : s_1 : s_2]\) are two points with \(t, s \in (\mathbb{C}^*)^2\). If we set \(g_i = \frac{s_i}{t_i}\), then \(g \cdot [1 : t_1 : t_2] = [1 : s_1 : s_2]\). Therefore, all of the points in the image of \(\phi_{P_1}\) are in the same torus orbit. This is exactly the set of points in \(\mathbb{P}^2\) with no coordinate equal to zero.

Since we are just multiplying each coordinate by a nonzero element of \(\mathbb{C}^*\), no point with a zero for a coordinate can be in the orbit above. Moreover, we see that the orbits of this action on \(\mathbb{P}^2\) must correspond to sets of points with fixed coordinates set to 0.

We can give a list of representatives for each orbit: \{\([1 : 1 : 1]\), \([1 : 1 : 0]\), \([1 : 0 : 1]\), \([0 : 1 : 1]\), \([1 : 0 : 0]\), \([0 : 1 : 0]\), \([0 : 0 : 1]\)\}. The closure of the first orbit is all of \(\mathbb{P}^2\), the closures of the next three orbits are the three coordinate axes. The last three points are torus-fixed points and each is the unique element in its orbit.

**Example 1.3** (The square \(P_4\)). As above, it is easy to see that setting \(g_i = \frac{s_i}{t_i}\) shows that the torus action is transitive on the image of \(\phi_{P_4}\).
The four points \([1 : 0 : 0 : 0], [0 : 1 : 0 : 0], [0 : 0 : 1 : 0], [0 : 0 : 0 : 1]\) are fixed because multiplying their coordinates by nonzero coordinates leaves them unchanged.

We also have \(g \cdot [1 : 1 : 0 : 0] = [1 : g_1 : 0 : 0], g \cdot [1 : 0 : 1 : 0] = [1 : 0 : g_2 : 0], g \cdot [0 : 1 : 0 : 1] = [0 : g_1 : 0 : g_1 g_2] = [0 : 1 : 0 : g_2]\) and \(g \cdot [0 : 0 : 1 : 1] = [0 : 0 : g_2 : g_1 g_2] = [0 : 0 : 1 : g_1]\).

To generalize what we have seen in the examples above, notice that there are as many torus-fixed points as vertices in each polygon, and as many torus-invariant curves as edges. (Of course, for surfaces, the number or edges is always equal to the number of vertices, but associating points to vertices, etc is what is correct in higher dimensions.) The decomposition of a polygon into its 2-dimensional face, edges, and vertices corresponds to the orbit-closure decomposition of the corresponding toric variety.

1.3. Limits. In this section, we answer the question: what do we get when we take the closure of these embeddings in projective space?

**Fact 1.4.** If \(P\) is a lattice polygon, then \(X_P\) is the union of the image of \(\phi_P\) together with a projective torus-invariant curve for each edge of \(P\). Two edges meet in a vertex in \(P\) if and only if the corresponding curves in \(X_P\) meet in a point that is fixed by the torus action.

In general we can use the torus action to find the limit points of \(X_P\). We will do this by parameterizing curves in \((\mathbb{C}^*)^2\) and taking their limits as the parameter goes to zero. The limit will be a point whose coordinates are all zero or 1. The orbit of such a point will either be the image of the torus, a dense subset of a torus-invariant curve, or a torus-fixed point. The 1-dimensional torus-orbits together with the torus-fixed points are the limit points that we add in when we take the closure of the image of \(\phi_P\).

The curves in \((\mathbb{C}^*)^2\) that we will use to find limit points have a special form.

**Definition 1.5** (pg. 37 in [1]). An integer vector \(v \in \mathbb{Z}^2\) corresponds to curve in \((\mathbb{C}^*)^2\) which sends \(z \in \mathbb{C}^*\) to

\[
\lambda_v(z) = (z^{v_1}, z^{v_2}).
\]

This curve is also a subgroup of \((\mathbb{C}^*)^2\) and we call it a 1-parameter subgroup.
Example 1.6 (The triangle $P_1$). We have $(t_1, t_2) \in (\mathbb{C}^*)^2$ mapping to $[1 : t_1 : t_2]$. 

\[
\begin{array}{c|c|c|c}
\text{v} & \lambda_v(z) & \phi_{P_1}(\lambda_v(z)) & \lim_{z \to 0} \phi_{P_1}(\lambda_v(z)) \\
(1, 1) & (z, z) & [1 : z : z] & [1 : 0 : 0] \\
(1, -1) & (z, \frac{1}{z}) & [1 : z : \frac{1}{z}] & [0 : 0 : 1] \\
(-1, 1) & (\frac{1}{z}, z) & [1 : \frac{1}{z} : z] & [0 : 1 : 0] \\
(1, 0) & (z, 1) & [1 : z : 1] & [1 : 0 : 1] \\
(0, 1) & (1, z) & [1 : 1 : z] & [1 : 1 : 0] \\
(-1, -1) & (\frac{1}{z}, \frac{1}{z}) & [1 : \frac{1}{z} : \frac{1}{z}] & [0 : 1 : 1] \\
\end{array}
\]

Notice that the first three limit points are fixed by the torus action. The last three points have orbits equal to $\mathbb{C}^*$. We obtain the closures of these orbits by adding in the appropriate fixed points to get projective lines.

Example 1.7 (The big triangle $P_2$). The lattice points in $P_2$ give the map $\phi_{P_2} : (\mathbb{C}^*)^2 \to \mathbb{P}^5$ sending 

\[
(t_1, t_2) \mapsto [1 : t_1 : t_2 : t_1^2 : t_2^2].
\]

Again, we make a table 

\[
\begin{array}{c|c|c|c}
\text{v} & \lambda_v(z) & \phi_{P_2}(\lambda_v(z)) & \lim_{z \to 0} \phi_{P_2}(\lambda_v(z)) \\
(1, 1) & (z, z) & [1 : z : z^2 : z^2 : z^2] & [1 : 0 : 0 : 0 : 0 : 0] \\
(1, -1) & (z, \frac{1}{z}) & [1 : z : \frac{1}{z} : 1 : z^2 : \frac{1}{z^2}] & [0 : 0 : 0 : 0 : 0 : 1] \\
(-1, 1) & (\frac{1}{z}, z) & [1 : \frac{1}{z} : z : 1 : \frac{1}{z^2} : z^2] & [0 : 0 : 0 : 0 : 1 : 0] \\
(1, 0) & (z, 1) & [1 : z : 1 : z^2 : 1] & [1 : 0 : 1 : 0 : 1 : 0] \\
(0, 1) & (1, z) & [1 : 1 : z : z : 1 : z^2] & [1 : 1 : 0 : 1 : 1 : 0] \\
(-1, -1) & (\frac{1}{z}, \frac{1}{z}) & [1 : \frac{1}{z} : \frac{1}{z^2} : \frac{1}{z^2} : \frac{1}{z^2}] & [0 : 0 : 0 : 1 : 1 : 1] \\
\end{array}
\]

The first three points correspond to fixed points. What happens to the last three under the torus action? We have 

\[
g \cdot [1 : 0 : 1 : 0 : 0 : 1] = [1 : 0 : g_2 : 0 : 0 : g_2^2],
\]

\[
g \cdot [1 : 1 : 0 : 0 : 1 : 0] = [1 : g_1 : 0 : 0 : g_1^2 : 0],
\]

\[
g \cdot [0 : 0 : 0 : 1 : 1 : 1] = [0 : 0 : 0 : g_1 g_2 : g_1^2 : g_2^2].
\]

What we see then, is that the closure of the image of $\phi_{P_2}$ contains three plane conics. In fact, what we are seeing is an embedding of $\mathbb{P}^2$ in which the three coordinate lines that we found earlier are all mapped to conics.
Note that the three edges in $P_1$ all had length one measured along the lattice, and in the limit we got three lines. The edges of $P_2$ all have lattice length 2, and in the limit we got three degree 2 curves. We see that $P_3$ has 3 edges of length 1 and one edge of length 2, so we expect to see limit points falling along three lines and one plane conic.

**Example 1.8** (The trapezoid $P_3$). For the trapezoid, we get the map $(t_1, t_2) \mapsto \left[ 1 : t_1 : t_2 : t_1 t_2 : t_1^2 \right]$. We find the torus invariant curves:

$$
\begin{array}{|c|c|c|c|}
\hline
v & \lambda_v(z) & \phi_{P_3}(\lambda_v(z)) & \lim_{z \to 0} \phi_{P_3}(\lambda_v(z)) = p \\
\hline
(1,0) & (z,1) & [1 : z : 1 : z^2] & [1 : 0 : 1 : 0] \\
(0,1) & (1,z) & [1 : 1 : z : z^2] & [1 : 1 : 0 : 0] \\
(-1,-1) & \left( \frac{1}{2}, \frac{1}{2} \right) & [1 : \frac{1}{2} : 1 : \frac{1}{2}] & [0 : 0 : 1 : 1] \\
(0,-1) & \left( 1, \frac{1}{z} \right) & [1 : 1 : \frac{1}{z} : \frac{1}{z^2}] & [0 : 0 : 1 : 0] \\
\hline
\end{array}
$$

We see that the orbit closures will be three projective lines and one plane conic.

**Example 1.9** (The square $P_4$). We have the map $\phi_{P_4} : (\mathbb{C}^*)^2 \to \mathbb{P}^3$ sending $t \mapsto [1 : t_1 : t_2 : t_1 t_2]$. We see that the orbit closures are 4 lines.

$$
\begin{array}{|c|c|c|c|}
\hline
v & \lambda_v(z) & \phi_{P_4}(\lambda_v(z)) & \lim_{z \to 0} \phi_{P_4}(\lambda_v(z)) = p \\
\hline
(1,0) & (z,1) & [1 : z : 1 : z] & [1 : 0 : 1 : 0] \\
(0,1) & (1,z) & [1 : 1 : z : z^2] & [1 : 1 : 0 : 0] \\
(-1,0) & \left( \frac{1}{2}, 1 \right) & [1 : \frac{1}{2} : 1 : \frac{1}{2}] & [0 : 1 : 0 : 1] \\
(0,-1) & \left( 1, \frac{1}{z} \right) & [1 : 1 : \frac{1}{z} : \frac{1}{z^2}] & [0 : 0 : 1 : 1] \\
\hline
\end{array}
$$

We see that the orbit closures are 4 lines.

Right now the choices of $v$ are a bit mysterious, but we will see later how to make sense of them.

1.4. **Smoothness I.**

**Fact 1.10.** If $P$ is a lattice polygon, then the singular points of $X_P$ are isolated points. This implies that if $p$ is a singular point its orbit must be zero-dimensional and hence that $p$ is a fixed point.

Suppose that $P$ is a lattice polygon and $v$ is a vertex which we may assume is at the origin in $\mathbb{R}^2$.

**Proposition 1.11.** If we can find an element of $GL_2(\mathbb{Z})$ that brings the edges incident at $v$ to the positive coordinate axes of $\mathbb{R}^2$, then $X_P$ is smooth at the fixed point corresponding to $v$. 

Proof. As $P$ lies in the first quadrant of $\mathbb{R}^2$, the fixed point corresponding to $v$ is the image of $(0, 0)$ in the natural extension of $\phi_P$ to $\mathbb{C}^2$. Moreover, $t_1$ and $t_2$ appear as coordinates of $\phi_P$. Therefore, $\frac{\partial \phi_P}{\partial t_1}|_{(0,0)}$ and $\frac{\partial \phi_P}{\partial t_2}|_{(0,0)}$ are linearly independent. These two vector span a 2-plane that is tangent to $X_P$ at $\phi_P(0, 0)$. Thus, the point is smooth. □

In the next lecture we will work on developing the local vocabulary to discuss smoothness algebraically and we will eventually see that the proposition above is an if and only if statement.

1.5. Exercises. The first few exercises will help you make the connection between our examples of toric varieties and classical examples.

(1) Substitute $t_1 = \frac{x_1}{x_0}$ and $t_2 = \frac{x_2}{x_0}$ into the map $\phi_{P_2}$. Homogenize the map to show that this allows us to identify $X_{P_2}$ with the image of a map $\nu_2 : \mathbb{P}^2 \rightarrow X_{P_2}$. The map $\nu_2$ is the quadratic Veronese embedding of $\mathbb{P}^2$ into $\mathbb{P}^5$. Show that $\nu_2$ is injective.

More generally, we have $d$-uple Veronese embeddings $\nu_d : \mathbb{P}^n \rightarrow \mathbb{P}^N$ whose coordinates may be given by a basis for the monomials in the homogeneous coordinates on $\mathbb{P}^n$ of degree $d$. What is $N$? Can you show that all of these maps are injective?

(2) We can define maps $\psi_i$ from $\mathbb{P}^2$ to $X_{P_i}$ where $i = 3, 4$ by deleting the appropriate coordinates from $\nu_2$. These maps are not defined on all of $\mathbb{P}^2$. Determine the points where each map is undefined.

(3) Consider the closure of the maps $\psi_i$ above. Can you identify the points that we add when we take the closure? Are the maps injective? If not, can you determine which points map to the same image?

(4) Notice that the map $\phi_{P_3}$ can be decomposed as

$$[1 : t_1 : t_2 : t_1 t_2 : t_1^2] = [1 : t_1 : 0 : 0 : t_1^2] + t_2[0 : 0 : 1 : t_1 : 0].$$

Substitute $t_2 = \frac{y_1}{y_0}$ into each summand. Convince yourself that $X_{P_3}$ has the following description: Map $\mathbb{P}^1$ into $\mathbb{P}^4$ simultaneously as a plane conic and a line where the line and plane do not meet. Then construct a surface by taking the union of all lines connecting the image of $p \in \mathbb{P}^1$ on the conic to its image on the line.

(5) Substitute $t_1 = \frac{x_1}{x_0}$ and $t_2 = \frac{y_1}{y_0}$ into the map $\phi_{P_4}$ and dehomogenize the map. Show that this gives an injective map from $\mathbb{P}^1 \times \mathbb{P}^1 = \{(x_0 : x_1, y_0 : y_1) | [x_0 : x_1] \in \mathbb{P}^1, [y_0 : y_1] \in \mathbb{P}^1\}$ to $X_{P_4}$. This is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^3$. Can you
give a natural description of the 1-dimensional orbit closures in $X_{\mathbb{P}_4}$ in terms of $\mathbb{P}^1 \times \mathbb{P}^1$?

In general, we have the Segre embeddings $\mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^N$ whose coordinates are all products of homogeneous coordinates for $\mathbb{P}^m$ with homogeneous coordinates for $\mathbb{P}^n$. What is $N$? Can you show that the Segre embeddings are injective?

(6) Find the torus fixed points of $X_{\mathbb{P}_i}$, $i = 3, 4$.

References


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