TORIC SURFACES
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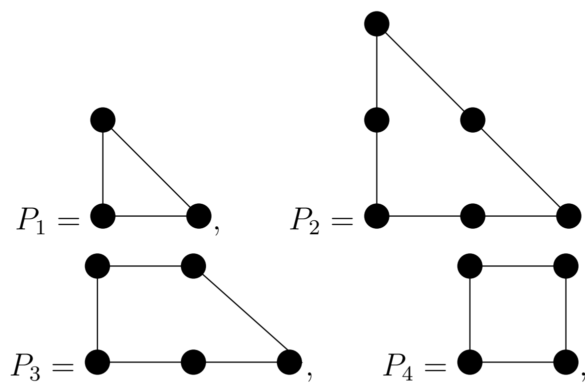
1. Day 2

In today's lecture you will learn how to

(1) realize several classical examples ($\mathbb{P}^1 \times \mathbb{P}^1$, the Veronese surface, $\mathbb{P}^2$, a rational normal scroll) as toric varieties using polygons.
(2) work with the action of the torus on X_P and compute the orbits of this action.
(3) visualize the limit points of our examples geometrically.
(4) determine if a toric surface is smooth by computing tangent planes.

1.1. **Examples.**

We give four examples of toric surfaces below. The varieties can be described via lattice polygons:



or via the corresponding integer matrices:

$$A_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 & 2 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}, A_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

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1.2. **The torus action.** In this section we will study the action of the *torus* $(\mathbb{C}^*)^d$ on a variety X_P and compute the orbits of the torus action. Restricting to the case of surfaces for simplicity, we will see that this action helps us to decompose the limit points of X_P into pieces that fit together in a nice way.

Recall the following definitions from group theory.

Definition 1.1. A group G *acts* on a set S if we have defined $g \cdot s \in S$ for every $g \in G$ and $s \in S$ so that

(1) $e \cdot s = s$, where $e \in G$ is the identity element.
(2) $(gh) \cdot s = g \cdot (h \cdot s))$.

Given an element $s \in S$, the *orbit* of s is

$$\{t \in S \mid \exists g \in G \text{ with } t = g \cdot s\}.$$

If s and t are in the same orbit, we write $s \equiv t$.

Given a lattice polytope P, we get an action of $T = (\mathbb{C}^*)^d$ on the image of ϕ_P. If $\mathbf{g} \in T$, define $\mathbf{g} \cdot \mathbf{t}^{m_i}$ by $\mathbf{g}^{m_i} \cdot \mathbf{t}^{m_i}$. Note that $\mathbf{g} \cdot \phi_P(\mathbf{t}) = \phi_P(g_1 t_1, \ldots, g_d t_d)$. This shows that T leaves the image of ϕ_P fixed. Clearly, this action also extends to projective space: $\mathbf{g} \cdot [x_0 : \cdots : x_n] = [\mathbf{g}^{\mathbf{m}_0} x_0 : \cdots : \mathbf{g}^{\mathbf{m}_n} x_n]$.

Let's look at examples 1 and 4.

Example 1.2 (The triangle P_1)**.** The set $P_1 \cap \mathbb{Z}^2 = \{(0,0), (1,0), (0,1)\}$. These lattice vectors give the map $\phi_{P_1} : (\mathbb{C}^*)^2 \to \mathbb{P}^2$ given by $\mathbf{t} \mapsto [1 : t_1 : t_2]$. The torus action is $\mathbf{g} \cdot [1 : t_1 : t_2] = [1 : g_1 t_1 : g_2 t_2]$.

First we'll check that the action is transitive on the image of ϕ_{P_1}. Indeed, suppose that $[1 : t_1 : t_2]$ and $[1 : s_1 : s_2]$ are two points with $\mathbf{t}, \mathbf{s} \in (\mathbb{C}^*)^2$. If we set $g_i = \frac{s_i}{t_i}$, then $\mathbf{g} \cdot [1 : t_1 : t_2] = [1 : s_1 : s_2]$. Therefore, all of the points in the image of ϕ_{P_1} are in the same torus orbit. This is exactly the set of points in $\mathbb{P}^2$ with no coordinate equal to zero.

Since we are just multiplying each coordinate by a nonzero element of $\mathbb{C}^*$, no point with a zero for a coordinate can be in the orbit above. Moreover, we see that the orbits of this action on $\mathbb{P}^2$ must correspond to sets of points with fixed coordinates set to 0.

We can give a list of representatives for each orbit: $\{[1 : 1 : 1], [1 : 1 : 0], [1 : 0 : 1], [0 : 1 : 1], [1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]\}$. The closure of the first orbit is all of $\mathbb{P}^2$, the closures of the next three orbits are the three coordinate axes. The last three points are torus-fixed points and each is the unique element in its orbit.

Example 1.3 (The square P_4)**.** As above, it is easy to see that setting $g_i = \frac{s_i}{t_i}$ shows that the torus action is transitive on the image of ϕ_{P_4}.

The four points $[1:0:0:0],[0:1:0:0],[0:0:1:0],[0:0:0:1]$ are fixed because multiplying their coordinates by nonzero coordinates leaves them unchanged.

We also have $\mathbf{g}\cdot[1:1:0:0]=[1:g_1:0:0]$, $\mathbf{g}\cdot[1:0:1:0]=[1:0:g_2:0]$, $\mathbf{g}\cdot[0:1:0:1]=[0:g_1:0:g_1g_2]=[0:1:0:g_2]$ and $\mathbf{g}\cdot[0:0:1:1]=[0:0:g_2:g_1g_2]=[0:0:1:g_1]$.

To generalize what we have seen in the examples above, notice that there are as many torus-fixed points as vertices in each polygon, and as many torus-invariant curves as edges. (Of course, for surfaces, the number or edges is always equal to the number of vertices, but associating points to vertices, etc is what is correct in higher dimensions.) The decomposition of a polygon into its 2-dimensional face, edges, and vertices corresponds to the orbit-closure decomposition of the corresponding toric variety.

1.3. **Limits.** In this section, we answer the question: what do we get when we take the closure of these embeddings in projective space?

Fact 1.4. If P is a lattice polygon, then X_P is the union of the image of ϕ_P together with a projective torus-invariant curve for each edge of P. Two edges meet in a vertex in P if and only if the corresponding curves in X_P meet in a point that is fixed by the torus action.

In general we can use the torus action to find the limit points of X_P. We will do this by parameterizing curves in $(\mathbb{C}^*)^2$ and taking their limits as the parameter goes to zero. The limit will be a point whose coordinates are all zero or 1. The orbit of such a point will either be the image of the torus, a dense subset of a torus-invariant curve, or a torus-fixed point. The 1-dimensional torus-orbits together with the torus-fixed points are the limit points that we add in when we take the closure of the image of ϕ_P.

The curves in $(\mathbb{C}^*)^2$ that we will use to find limit points have a special form.

Definition 1.5 (pg. 37 in [1])**.** An integer vector $\mathbf{v}\in\mathbb{Z}^2$ corresponds to curve in $(\mathbb{C}^*)^2$ which sends $z\in\mathbb{C}^*$ to

$$\lambda_{\mathbf{v}}(z)=(z^{v_1},z^{v_2}).$$

This curve is also a subgroup of $(\mathbb{C}^*)^2$ and we call it a *1-parameter subgroup.*

Example 1.6 (The triangle P_1)**.** We have $(t_1, t_2) \in (\mathbb{C}^*)^2$ mapping to $[1 : t_1 : t_2]$.

$\mathbf{v}$	$\lambda_{\mathbf{v}}(z)$	$\phi_{P_1}(\lambda_{\mathbf{v}}(z))$	$\lim_{z\to 0} \phi_{P_1}(\lambda_{\mathbf{v}}(z))$
$(1,1)$	(z,z)	$[1:z:z]$	$[1:0:0]$
$(1,-1)$	$(z,\frac{1}{z})$	$[1:z:\frac{1}{z}]$	$[0:0:1]$
$(-1,1)$	$(\frac{1}{z},z)$	$[1:\frac{1}{z}:z]$	$[0:1:0]$
$(1,0)$	$(z,1)$	$[1:z:1]$	$[1:0:1]$
$(0,1)$	$(1,z)$	$[1:1:z]$	$[1:1:0]$
$(-1,-1)$	$(\frac{1}{z},\frac{1}{z})$	$[1:\frac{1}{z}:\frac{1}{z}]$	$[0:1:1]$

Notice that the first three limit points are fixed by the torus action. The last three points have orbits equal to $\mathbb{C}^*$. We obtain the closures of these orbits by adding in the appropriate fixed points to get projective lines.

Example 1.7 (The big triangle P_2)**.** The lattice points in P_2 give the map $\phi_{P_2} : (\mathbb{C}^*)^2 \to \mathbb{P}^5$ sending

$$(t_1, t_2) \mapsto [1 : t_1 : t_2 : t_1t_2 : t_1^2 : t_2^2].$$

Again, we make a table

$\mathbf{v}$	$\lambda_{\mathbf{v}}(z)$	$\phi_{P_2}(\lambda_{\mathbf{v}}(z))$	$\lim_{z\to 0} \phi_{P_2}(\lambda_{\mathbf{v}}(z))$
$(1,1)$	(z,z)	$[1:z:z:z^2:z^2:z^2]$	$[1:0:0:0:0:0]$
$(1,-1)$	$(z,\frac{1}{z})$	$[1:z:\frac{1}{z}:1:z^2:\frac{1}{z^2}]$	$[0:0:0:0:0:1]$
$(-1,1)$	$(\frac{1}{z},z)$	$[1:\frac{1}{z}:z:1:\frac{1}{z^2}:z^2]$	$[0:0:0:0:1:0]$
$(1,0)$	$(z,1)$	$[1:z:1:z:z^2:1]$	$[1:0:1:0:0:1]$
$(0,1)$	$(1,z)$	$[1:1:z:z:1:z^2]$	$[1:1:0:0:1:0]$
$(-1,-1)$	$(\frac{1}{z},\frac{1}{z})$	$[1:\frac{1}{z}:\frac{1}{z}:\frac{1}{z^2}:\frac{1}{z^2}:\frac{1}{z^2}]$	$[0:0:0:1:1:1]$

The first three points correspond to fixed points. What happens to the last three under the torus action? We have

$$\mathbf{g} \cdot [1:0:1:0:0:1] = [1:0:g_2:0:0:g_2^2],$$

$$\mathbf{g} \cdot [1:1:0:0:1:0] = [1:g_1:0:0:g_1^2:0],$$

$$\mathbf{g} \cdot [0:0:0:1:1:1] = [0:0:0:g_1g_2:g_1^2:g_2^2].$$

What we see then, is that the closure of the image of ϕ_{P_2} contains three plane conics. In fact, what we are seeing is an embedding of $\mathbb{P}^2$ in which the three coordinate lines that we found earlier are all mapped to conics.

Note that the three edges in P_1 all had length one measured along the lattice, and in the limit we got three lines. The edges of P_2 all have lattice length 2, and in the limit we got three degree 2 curves. We see that P_3 has 3 edges of length 1 and one edge of length 2, so we expect to see limit points falling along three lines and one plane conic.

Example 1.8 (The trapezoid P_3)**.** For the trapezoid, we get the map $(t_1, t_2) \mapsto [1 : t_1 : t_2 : t_1t_2 : t_1^2]$. We find the torus invariant curves:

$\mathbf{v}$	$\lambda_{\mathbf{v}}(z)$	$\phi_{P_3}(\lambda_{\mathbf{v}}(z))$	$\lim_{z\to 0} \phi_{P_3}(\lambda_{\mathbf{v}}(z)) = p$	$\mathbf{g} \cdot p$
$(1, 0)$	$(z, 1)$	$[1 : z : 1 : z : z^2]$	$[1 : 0 : 1 : 0 : 0]$	$[1 : 0 : g_2 : 0 : 0]$
$(0, 1)$	$(1, z)$	$[1 : 1 : z : z : 1]$	$[1 : 1 : 0 : 0 : 1]$	$[1 : g_1 : 0 : 0 : g_1^2]$
$(-1, -1)$	$(\frac{1}{z}, \frac{1}{z})$	$[1 : \frac{1}{z} : \frac{1}{z} : \frac{1}{z^2} : \frac{1}{z^2}]$	$[0 : 0 : 0 : 1 : 1]$	$[0 : 0 : 0 : g_2 : g_1]$
$(0, -1)$	$(1, \frac{1}{z})$	$[1 : 1 : \frac{1}{z} : \frac{1}{z} : 1]$	$[0 : 0 : 1 : 1 : 0]$	$[0 : 0 : 1 : g_1 : 0]$

We see that the orbit closures will be three projective lines and one plane conic.

Example 1.9 (The square P_4)**.** We have the map $\phi_{P_4} : (\mathbb{C}^*)^2 \to \mathbb{P}^3$ sending $\mathbf{t} \mapsto [1 : t_1 : t_2 : t_1t_2]$.

$\mathbf{v}$	$\lambda_{\mathbf{v}}(z)$	$\phi_{P_4}(\lambda_{\mathbf{v}}(z))$	$\lim_{z\to 0} \phi_{P_4}(\lambda_{\mathbf{v}}(z))$	$\mathbf{g} \cdot p$
$(1, 0)$	$(z, 1)$	$[1 : z : 1 : z]$	$[1 : 0 : 1 : 0]$	$[1 : 0 : g_2 : 0]$
$(0, 1)$	$(1, z)$	$[1 : 1 : z : z]$	$[1 : 1 : 0 : 0]$	$[1 : g_1 : 0 : 0]$
$(-1, 0)$	$(\frac{1}{z}, 1)$	$[1 : \frac{1}{z} : 1 : \frac{1}{z}]$	$[0 : 1 : 0 : 1]$	$[0 : 1 : 0 : g_2]$
$(0, -1)$	$(1, \frac{1}{z})$	$[1 : 1 : \frac{1}{z} : \frac{1}{z}]$	$[0 : 0 : 1 : 1]$	$[0 : 0 : 1 : g_1]$

We see that the orbit closures are 4 lines.

Right now the choices of $\mathbf{v}$ are a bit mysterious, but we will see later how to make sense of them.

1.4. **Smoothness I.**

Fact 1.10. If P is a lattice polygon, then the singular points of X_P are isolated points. This implies that if p is a singular point its orbit must be zero-dimensional and hence that p is a fixed point.

Suppose that P is a lattice polygon and v is a vertex which we may assume is at the origin in $\mathbb{R}^2$.

Proposition 1.11. *If we can find an element of $GL_2(\mathbb{Z})$ that brings the edges incident at v to the positive coordinate axes of $\mathbb{R}^2$, then X_P is smooth at the fixed point corresponding to v.*

Proof. As P lies in the first quadrant of $\mathbb{R}^2$, the fixed point corresponding to v is the image of $(0,0)$ in the natural extension of ϕ_P to $\mathbb{C}^2$. Moreover, t_1 and t_2 appear as coordinates of ϕ_P. Therefore, $\frac{\partial \phi_P}{\partial t_1}|_{(0,0)}$ and $\frac{\partial \phi_P}{\partial t_2}|_{(0,0)}$ are linearly independent. These two vector span a 2-plane that is tangent to X_P at $\phi_P(0,0)$. Thus, the point is smooth. □

In the next lecture we will work on developing the local vocabulary to discuss smoothness algebraically and we will eventually see that the proposition above is an if and only if statement.

1.5. **Exercises.** The first few exercises will help you make the connection between our examples of toric varieties and classical examples.

(1) Substitute $t_1 = \frac{x_1}{x_0}$ and $t_2 = \frac{x_2}{x_0}$ into the map ϕ_{P_2}. Homogenize the map to show that this allows us to identify X_{P_2} with the image of a map $\nu_2 : \mathbb{P}^2 \to X_{P_2}$. The map ν_2 is the quadratic *Veronese* emebdding of $\mathbb{P}^2$ into $\mathbb{P}^5$. Show that ν_2 is injective.

More generally, we have d-uple Veronese embeddings $\nu_d : \mathbb{P}^n \to \mathbb{P}^N$ whose coordinates may be given by a basis for the monomials in the homogeneous coordinates on $\mathbb{P}^n$ of degree d. What is N? Can you show that all of these maps are injective?

(2) We can define maps ψ_i from $\mathbb{P}^2$ to X_{P_i} where $i = 3, 4$ by deleting the appropriate coordinates from ν_2. These maps are not defined on all of $\mathbb{P}^2$. Determine the points where the each map is undefined.

(3) Consider the closure of the maps ψ_i above. Can you identify the points that we add when we take the closure? Are the maps injective? If not, can you determine which points map to the same image?

(4) Notice that the map ϕ_{P_3} can be decomposed as

$$[1 : t_1 : t_2 : t_1t_2 : t_1^2] = [1 : t_1 : 0 : 0 : t_1^2] + t_2[0 : 0 : 1 : t_1 : 0].$$

Substitute $t_2 = \frac{y_1}{y_0}$ into each summand. Convince yourself that X_{P_3} has the following description: Map $\mathbb{P}^1$ into $\mathbb{P}^4$ simultaneously as a plane conic and a line where the line and plane do not meet. Then construct a surface by taking the union of all lines connecting the image of $p \in \mathbb{P}^1$ on the conic to its image on the line.

(5) Substitute $t_1 = \frac{x_1}{x_0}$ and $t_2 = \frac{y_1}{y_0}$ into the map ϕ_{P_4} and dehomogenize the map. Show that this gives an injective map from $\mathbb{P}^1 \times \mathbb{P}^1 = \{([x_0 : x_1], [y_0 : y_1]) \mid [x_0 : x_1] \in \mathbb{P}^1,\ [y_0 : y_1]) \in \mathbb{P}^1\}$ to X_{P_4}. This is the *Segre embedding* of $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$. Can you

give a natural description of the 1-dimensional orbit closures in X_{P_4} in terms of $\mathbb{P}^1 \times \mathbb{P}^1$?

In general, we have the Segre embeddings $\mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^N$ whose coordinates are all products of homgeneous coordinates for $\mathbb{P}^m$ with homogeneous coordinates for $\mathbb{P}^n$. What is N? Can you show that the Segre embeddings are injective?

(6) Find the torus fixed points of X_{P_i}, $i = 3, 4$.

References

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