

TORIC SURFACES
PCMI UNDERGRADUATE COURSE, JULY 2008

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1. DAY 3

Our goal for today is to identify the singular points of X_P where P is a lattice polygon. The plan is as follows. Suppose $\mathbf{p} \in X_P$ is a fixed point. Find an open set containing \mathbf{p} of the form $X_A \subseteq \mathbb{C}^n$ so that $\mathbf{p} = \mathbf{0}$. Use the lattice to compute $\dim_{\mathbb{C}} m_{\mathbf{p}}/m_{\mathbf{p}}^2$.

To accomplish our goal we will learn how to

- (1) find an open affine covering of X_P by affine toric varieties.
- (2) relate our affine covering to the inner normal fan of P .

Observation 1.1. Suppose that $P \subset \mathbb{R}^2$ is a lattice polygon with $P \cap \mathbb{Z}^2 = \{\mathbf{a}_0 = \mathbf{0}, \mathbf{a}_1, \dots, \mathbf{a}_n\}$. Then $\phi_P(\mathbf{t}) = [1 : \mathbf{t}^{\mathbf{a}_1} : \dots : \mathbf{t}^{\mathbf{a}_n}]$ maps to $U_0 = \mathbb{C}^n$.

Let $A = (\mathbf{a}_1 \cdots \mathbf{a}_n)$. Then

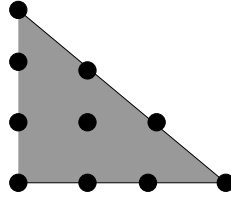
- (1) $X_A = X_P \cap U_0$ which is an open subset of X_P .
- (2) The torus-fixed point corresponding to \mathbf{a}_0 is $\mathbf{0} \in X_A$.
- (3) $\mathbb{C}[x_1, \dots, x_n]/I_A \cong \mathbb{C}[\mathbf{t}^{\mathbf{a}_1}, \dots, \mathbf{t}^{\mathbf{a}_n}]$.

Example 1.2. Let $Q_0 = \text{conv}\{(0,0), (1,0), (0,1)\}$ be the standard simplex in the plane. We get the map $\phi_{Q_0}(\mathbf{t}) = [1 : t_1 : t_2]$ which maps to U_0 . To see the affine variety $X_{Q_0} \cap U_0$, as above let $A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $\phi_{A_0}(\mathbf{t}) = (t_1, t_2)$, and we see that $X_{A_0} = \mathbb{C}^2$. In this case $\mathbb{C}[t_1, t_2] \cong \mathbb{C}[x, y]$ is just a polynomial ring. Moreover, note that the exponent vectors of monomials in $\mathbb{C}[t_1, t_2]$ are precisely the lattice vectors in the first quadrant.

Now consider $Q_1 = \text{conv}\{(-1,0), (0,0), (-1,1)\}$, which is just a translation of Q_0 . This gives rise to the map $\phi_{Q_1}(\mathbf{t}) = [\frac{1}{t_1} : 1 : \frac{t_2}{t_1}]$. Of course, $[\frac{1}{t_1} : 1 : \frac{t_2}{t_1}] = [1 : t_1 : t_2]$ if $t_1 \neq 0$, so then $X_{Q_0} = X_{Q_1}$ even though ϕ_{Q_1} maps $(\mathbb{C}^*)^2$ to U_1 . The affine variety that we see is again

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\mathbb{C}^2 , although in this case $A_1 = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$ and $\mathbb{C}[\frac{1}{t_1}, \frac{t_2}{t_1}] \cong \mathbb{C}[x, y]$ has exponent vectors contained in the cone below.



The affine toric variety gotten from translating the top vertex of the triangle to the origin can be analyzed in a similar fashion.

We introduce language that will allow us to discuss the example above more precisely.

Definition 1.3. A finite set of vectors $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\} \subset \mathbb{R}^d$ generates a *convex polyhedral cone*

$$\text{cone } \mathcal{A} = \{\lambda_1 \mathbf{a}_1 + \dots + \lambda_n \mathbf{a}_n \mid \lambda_i \in \mathbb{R}_{\geq 0}\}.$$

The dimension of a cone is the dimension of its linear span. A cone is *rational* if it can be generated by lattice vectors. We will simply write “cone” in these notes but will always mean a rational convex polyhedral cone. We say that the cone is *strongly convex* if it doesn’t contain \mathbf{v} and $-\mathbf{v}$ for any $\mathbf{v} \in \mathbb{R}^d$. (See Proposition 3 on pg. 14 of [1].)

Definition 1.4. If $\sigma \subset \mathbb{R}^d$ is a cone, we can define the ring $\mathbb{C}[\mathbf{t}^{\mathbf{a}} \mid \mathbf{a} \in \mathbb{Z}^d \cap \sigma]$. In other words, we have defined a ring that is generated by monomials whose exponent vectors lie in the cone σ .

Assumption 1.5. Assume that \mathcal{A} generates the set $(\text{cone } \mathcal{A}) \cap \mathbb{Z}^d$ as a semigroup.

Fact 1.6. Let $P \subset \mathbb{R}^d$ be a lattice polygon with $P \cap \mathbb{Z}^d = \{\mathbf{a}_0 = \mathbf{0}, \dots, \mathbf{a}_n\}$. If $\sigma = \text{cone}\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ and $A = (\mathbf{a}_1 \cdots \mathbf{a}_n)$, then $\mathbb{C}[\mathbf{t}^{\mathbf{a}_1}, \dots, \mathbf{t}^{\mathbf{a}_n}] \cong \mathbb{C}[x_1, \dots, x_n]/I_A$.

1.1. Smoothness II. Suppose we want to investigate whether the point at the origin in an affine variety X_A is singular. In $\mathbb{C}[x_1, \dots, x_n]/I_A$, the ideal corresponding to the origin is $m = \langle x_1, \dots, x_n \rangle$. (Of course, although x_1, \dots, x_n form a set of minimal generators for the ideal of the origin in \mathbb{C}^n , they may not be minimal generators of the ideal they generate in the quotient ring.)

In the theorem below we translate the computation of $\dim_{\mathbb{C}} m/m^2$ into terminology using only the lattice.

Theorem 1.7 (pgs. 28-29 in [1]). *Let A be a $d \times n$ integer matrix and σ be the cone generated by the columns of A . Assume that the columns of A generate the set $\sigma \cap \mathbb{Z}^d$ as a semigroup, i.e., that every element of $\sigma \cap \mathbb{Z}^d$ is a finite sum of some subset of the columns of A .*

The point $\mathbf{0} \in X_A$ is smooth if and only if the dimension of X_A is equal to the size of the set

$$\{\mathbf{v} \in (\sigma \cap \mathbb{Z}^d) \mid \mathbf{v} \neq \mathbf{0}, \mathbf{v} \neq \mathbf{u} + \mathbf{w}, \mathbf{u}, \mathbf{w} \in (\sigma \cap \mathbb{Z}^d) \setminus \{\mathbf{0}\}\}.$$

Proof. Let $m_{\mathbf{0}}$ denote the ideal of the origin in $\mathbb{C}[\mathbf{t}^{\mathbf{a}_1}, \dots, \mathbf{t}^{\mathbf{a}_n}] \cong \mathbb{C}[x_1, \dots, x_n]/I_A$. Note that $m_{\mathbf{0}}$ is spanned by monomials with exponents in $(\sigma \cap \mathbb{Z}^d) \setminus \{\mathbf{0}\}$. The ideal $m_{\mathbf{0}}^2$ is spanned by products of monomials in $m_{\mathbf{0}}$. Hence, the exponent vectors of elements of $m_{\mathbf{0}}^2$ are precisely the set of elements in $(\sigma \cap \mathbb{Z}^d) \setminus \{\mathbf{0}\}$ that are the nontrivial sum of two other such elements.

Therefore, to compute the vector space dimension of $m_{\mathbf{0}}/m_{\mathbf{0}}^2$, we just need to count the vectors in $(\sigma \cap \mathbb{Z}^d) \setminus \{\mathbf{0}\}$ that cannot be written as the nontrivial sum of two other elements in $(\sigma \cap \mathbb{Z}^d) \setminus \{\mathbf{0}\}$. By definition the variety X_A is smooth at $\mathbf{0}$ if and only if $\dim X_A = \dim_{\mathbb{C}} m_{\mathbf{0}}/m_{\mathbf{0}}^2$, so we are done. \square

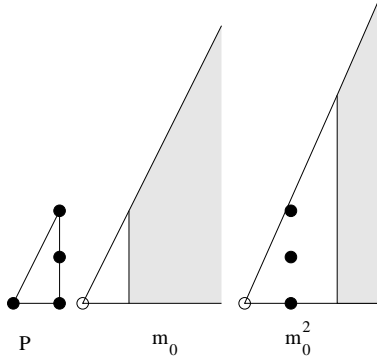
Fact 1.8. If $\sigma \subset \mathbb{R}^d$ is a d -dimensional cone, then $\sigma \cap \mathbb{Z}^d$ generates \mathbb{Z}^d as a group. Essentially this is because there are no “holes” in the cone σ . For a converse to this see the Exercise on pg. 19 of [1].

Corollary 1.9 (Proposition on pg. 29 of [1]). *The variety X_A is smooth at $\mathbf{0}$ if and only if the first lattice vectors along the rays of σ are a \mathbb{Z} -basis for \mathbb{Z}^d .*

Proof. The vector space $m_{\mathbf{0}}/m_{\mathbf{0}}^2$ has a basis consisting of monomials, and this basis must contain the first lattice vectors along the rays of σ . If σ is d -dimensional, it must have at least d generators. This vector space has dimension $d = \dim X_A$ if and only if $\mathbf{0}$ is a smooth point. Therefore, $\mathbf{0}$ is a smooth point if and only if $m_{\mathbf{0}}/m_{\mathbf{0}}^2$ has a basis consisting of the first lattice vectors along the rays of σ . These vectors clearly generate $\sigma \cap \mathbb{Z}^d$ if all of the elements of $m_{\mathbf{0}}^2$ are the sum of at least two of them. Since $\sigma \cap \mathbb{Z}^d$ generates \mathbb{Z}^d as a group, these lattice vectors must also generate \mathbb{Z}^d as a group. \square

Here is an example of a singular point on a toric surface.

Example 1.10 (Cone over a conic). Let P be the convex hull of $\{(0, 0), (1, 0), (1, 2)\}$ depicted below.



We see the region containing the monomials in $m_{\mathbf{0}}$ and the region containing the monomials in $m_{\mathbf{0}}^2$. The three black dots in the cone on the right represent the monomials in $m_{\mathbf{0}}$ that are not in $m_{\mathbf{0}}^2$. We see that $\dim_{\mathbb{C}} m_{\mathbf{0}}/m_{\mathbf{0}}^2 = 3 \neq \dim X_P = 2$. Therefore we see that the fixed point corresponding to $\mathbf{0}$ is a singular point.

1.1.1. *Keeping track of cones: fans and duals.*

Example 1.11 (X_{P_3}). Suppose we try to draw the cone at each vertex of P_3 . Notice that we can't draw all of them in the same picture without overlaps.

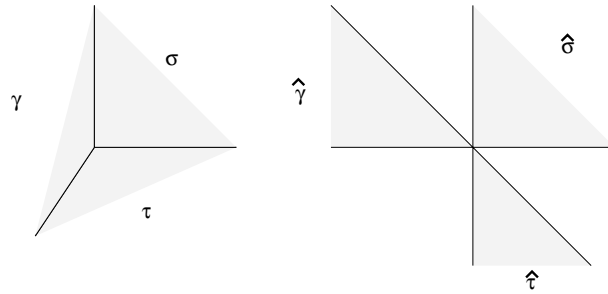
There is a *dual* picture that involves all of the same data, but gives us better intuition about how the different affine patches interact with each other.

Definition 1.12. If $\sigma \subset \mathbb{R}^d$ is a cone, then its *dual* is defined to be

$$\hat{\sigma} = \{\mathbf{u} \in \mathbb{R}^d \mid \mathbf{u} \cdot \mathbf{v} \geq 0, \forall \mathbf{v} \in \sigma\}.$$

Example 1.13 (Dual cones for P_1). On the right we have depicted the cones that we get when we translate each vertex of P_1 to the origin. Their duals are on the left. Notice that the dual cones fit together nicely. Note that we denote the cones on the left by σ_i and the cones on the right by $\hat{\sigma}_i$, which might seem odd from our point of view so far. However, in the literature one typically starts with the picture on the left and then dualizes to produce the picture on the right. Since the dual of the dual of a cone is the original cone, the pictures convey equivalent data, and we follow the standard conventions to ease the

transition to other readings.



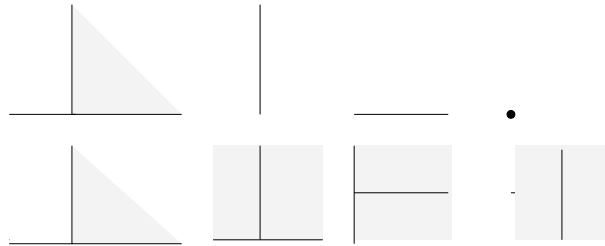
Observe that the rays of the dual cones are exactly the normal vectors to the edges of the polygon. The definitions below give us precise language for describing how these cones fit together.

Definition 1.14. If $\sigma \subset \mathbb{R}^d$ is a cone, we define a *face* of σ to be a set of the form

$$\{\mathbf{v} \in \sigma \mid \mathbf{u} \cdot \mathbf{v} = 0\}$$

for some $\mathbf{u} \in \hat{\sigma}$.

Example 1.15 (Faces of some cone). We have depicted the cone generated by $(1, 0), (0, 1)$ together with each of its faces. Below each face we give its dual.



Definition 1.16. A *fan* $\Delta \subset \mathbb{R}^d$ is a collection of cones satisfying

- (1) If $\sigma \in \Delta$ and τ is a face of σ , then $\tau \in \Delta$.
- (2) If $\sigma, \tau \in \Delta$, then $\sigma \cap \tau$ is simultaneously a face of σ and a face of τ .

Definition 1.17. If $P \subset \mathbb{R}^d$ is a polytope, its *inner normal fan* is the fan whose maximal cones are the duals of the cones at each vertex of P .

If $P \subset \mathbb{R}^2$ is a lattice polygon, then we can get its inner normal fan as follows. For each edge e of P , translate the normal vector pointing into P to the origin and let ρ_e be the ray spanned by this vector. These rays are the rays of the inner normal fan of P .

1.2. Exercises.

- (1) Show that the other two fixed points in the variety X_P given in Example 1.10 are smooth points. Can you do this by computing $\dim m/m^2$ and also by proving that the first lattice points on the vectors pointing away from the vertices of P form a basis for \mathbb{Z} ?
- (2) Draw the inner normal fans for each of the polygons that we have seen so far.
- (3) Pick a fan above. Draw the dual of each cone in the fan.
- (4) Blowup warmup: Let $B \subset \mathbb{C}^2 \times \mathbb{P}^1$ given by the equation $x_0y_1 = x_1y_0$ be the blowup of \mathbb{C}^2 at $\mathbf{0}$ with projection map $\pi : B \rightarrow \mathbb{C}^2$. When we blowup we replace $\mathbf{0}$ with a copy of \mathbb{P}^1 which we call the *exceptional divisor*. Moreover, the lines through $\mathbf{0}$ get separated in the blowup and instead of intersecting each other at $\mathbf{0}$, each now intersects the new copy of \mathbb{P}^1 , at a point corresponding to its slope. We can see this explicitly as follows:
 - (a) Let $x_1 - mx_0 = 0$ be the equation of a line with nonzero slope. Recall that U_0 is the coordinate patch where $y_0 = 1$. Use the equation of the blowup to rewrite the equation of the line in the coordinates for U_0 .
 - (b) The new equation should factor in U_0 . What does this tell us geometrically about the zero set of this equation in U_0 ?
 - (c) Use the coordinates of U_0 to write down the points of π^{-1} of the line in \mathbb{C}^2 defined by $x_1 - mx_0 = 0$. You should see that the inverse image “remembers” the slope.
 - (d) The closure of the inverse image of the nonzero points on your line is its *strict transform*. Find the coordinates of the intersection of the strict transform of your line with the exceptional divisor.

2. COMPUTING WITH MACAULAY 2

Research mathematicians often use computer software to investigate examples that help them create and test conjectures. In algebraic geometry and commutative algebra there are several software packages designed for mathematicians by mathematicians. *Macaulay 2* is on the Macs in the lab. If you'd like to get a feel for the kind of software that research mathematicians use, you can come down to the computer lab and play with the ideals from day 1 using *Macaulay 2*. You can run *Macaulay 2* by typing “M2” (without the quotation marks) in a terminal window.

Macaulay 2 code for computing toric ideals from an integer matrix A is given by Hosten and Smith on pg. 87 of "Computations in Algebraic Geometry with Macaulay 2", David Eisenbud, Daniel Grayson, Michael Stillman, Bernd Sturmfels, eds., Springer-Verlag, 2002.

The code below shows you how to load a file containing the binomial ideal code into Macaulay 2 and how to use it to find the binomial ideal corresponding to an integer matrix. The load command assumes that the file "toricIdeal.m2" is in the same directory from which you called Macaulay 2.

```
i1 : load "toricIdeal.m2"

i2 : A = matrix{{1,1,1,1},{0,1,2,3}}

o2 = | 1 1 1 1 |
      | 0 1 2 3 |

      2      4
o2 : Matrix ZZ <---- ZZ

i3 : toricIdeal(A)
```

I'll put the file "toricIdeal.m2" on the webpage so you can download it. You can either put it in your working directory or you can cut and paste the code into *Macaulay 2* line by line.

The *Macaulay 2* webpage is at <http://www.math.uiuc.edu/Macaulay2/>.

REFERENCES

- [1] William Fulton, *Introduction to Toric Varieties*, Annals of Math. studies no. 131, Princeton Univ. Press, Princeton, 1993.
- [2] Israel M. Gelfand, Mikhail M. Kapranov, and Andrei V. Zelvinsky, *Discriminants, Resultants and Multidimensional Determinants*, Birkhäuser, Boston, 1994.
- [3] Bernd Sturmfels, *Gröbner Bases and Convex Polytopes*, Univ. Lecture Series, v. 8, Amer. Math. Soc. , Providence, 1996.

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