

TORIC SURFACES
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1. DAY 4

Recall 1.1. Suppose that $P \subset \mathbb{R}^d$ is a lattice polytope with vertex $\mathbf{a}_0 = \mathbf{0}$. Let σ be the dual of the cone generated by P . The lattice points in the cone generated by the lattice points in P are the exponent vectors of Laurent monomials in $A_\sigma \cong \mathbb{C}[\mathbf{t}^{\mathbf{a}_1}, \dots, \mathbf{t}^{\mathbf{a}_n}] \cong \mathbb{C}[x_1, \dots, x_n]/I_A$.

Remark 1.2. Since we are working in the dimension two today, and we are not going to be looking at the equations of our toric varieties, we will let our semigroup algebras be subsets of $S[x, y, x^{-1}, y^{-1}]$.

Observation 1.3. The ring A_σ corresponds to an “abstract” affine toric variety U_σ . (See section 1.3 of [1] for a treatment of toric varieties which begins with this point of view.) In the setup above, $X_A \subset \mathbb{C}^n$ is an embedding of U_σ into affine space.

Goals:

- (1) Understand how a fan Δ tells us how varieties U_σ fit together to give an abstract toric variety X_Δ .
- (2) Understand how subdivisions of cones give maps of toric varieties.
- (3) Use (1) and (2) to resolve the singularities of a toric surface!

1.1. Fans and duals II. In this section we will see that the dual picture is useful not just because the cones fit together nicely. The point is that faces of the dual cones correspond algebraically to the coordinate rings of open subsets in a nice way. We follow the notation of [1] as closely as possible to help students who wish to transition from these notes to the other text.

Assumption 1.4. In what follows today we will assume that $d = 2$ throughout. This is the situation that we will depict pictorially. There are natural generalizations for arbitrary d , with the added complication

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that if $d \geq 3$ specifying a collection of rays does not automatically specify a complete fan.

Since we will be moving between cones and their duals frequently, it will be useful to introduce notation that will help us to remember that a cone and its dual ought to live in different spaces.

Definition 1.5. As in [1], let $M_{\mathbb{R}}$ denote the copy of \mathbb{R}^2 that contains P (and the cones at the vertices of P). We let $N_{\mathbb{R}}$ denote the copy of \mathbb{R}^2 containing the inner normal fan of P . We denote the lattice in $M_{\mathbb{R}}$ by M and write N for the lattice in $N_{\mathbb{R}}$.

Although A_{σ} can be realized as $\mathbb{C}[x_1, \dots, x_n]/I_A$ (in many different ways, in fact), it is a ring in its own right independent of its realization as a quotient of a polynomial ring. In fact, A_{σ} corresponds to an “abstract” affine toric variety $U_{\sigma} \cong X_A$ via its *spectrum* of prime ideals. For our purposes, we will think only of the set of all maximal ideals of A_{σ} , for these ideals correspond to points of U_{σ} in the usual sense (closed points in the language of algebraic geometry). We will denote this abstract affine toric variety by $\text{Specm } A_{\sigma}$, following pg. 14 of [1].

Our first goal today is to understand how to think about these abstract affine toric varieties U_{σ} and how a fan tells us how they glue together to give an arbitrary toric variety.

Here are some useful facts.

- Lemma 1.6.**
- (1) *If $\tau \subset \sigma \subset N$, then $\hat{\sigma} \subset \hat{\tau}$ and $A_{\sigma} \hookrightarrow A_{\tau}$.*
 - (2) *If τ is a face of σ , then A_{τ} is gotten from A_{σ} by adjoining the inverse of a monomial in A_{σ} .*
 - (3) *The maximal ideals of A_{τ} correspond precisely to the maximal ideals of A_{σ} which do not contain the monomial that is inverted.*

Proof. We leave part (1) as an exercise. for part (2), see Proposition 2 on pg. 13 of [1]. Part (3) requires a bit of commutative algebra. Let $f \in A_{\sigma}$ be the monomial that we invert to get A_{τ} . If $m \subset A_{\tau}$ is maximal, let \bar{m} be its intersection with A_{σ} . If \bar{m} is not maximal, then it is contained in some maximal ideal n . If n does not contain f , then its expansion to A_{τ} is a nontrivial ideal strictly containing m , which contradicts the maximality of m . Therefore, every maximal ideal containing \bar{m} must contain f . But, A_{σ} is a Jacobson ring, which means that any prime ideal is the intersection of all of the maximal ideals containing it. As \bar{m} must be prime, this would imply $f \in \bar{m}$, which is a contradiction because then \bar{m} would contain 1. \square

Example 1.7 (Cone for \mathbb{C}^2). Let $\sigma = \text{cone}\{e_1, e_2\} \subset N_{\mathbb{R}}$ and note that it is its own dual. The vectors in $\hat{\sigma} \subset M_{\mathbb{R}}$ correspond to the monomials in $\mathbb{C}[x, y]$. This is the coordinate ring of \mathbb{C}^2 .

What do the faces of σ correspond to? The face cone $\{e_1\}$ has dual equal to the right half-plane. The corresponding algebra is $\mathbb{C}[x, y, \frac{1}{y}]$. This is the ring of polynomial functions on $\mathbb{C}^2 \setminus \{(x, 0) \mid x \in \mathbb{C}\}$. Similarly, the face cone $\{e_2\}$ corresponds to $\mathbb{C}[x, y, \frac{1}{x}]$ which is the ring of polynomial functions on \mathbb{C}^2 minus the y -axis.

1.2. Limit points and the fan. Suppose that P is a lattice polygon. How do we find the limit points corresponding to the faces of P ?

Let Δ be the inner normal fan of P . Each face of P corresponds to a cone in Δ in a natural way. The maximal face of P corresponds to the origin. If e is an edge of P , it corresponds to the 1-dimensional cone spanned by an inner normal vector to e . If v is a vertex of P it corresponds to the 2-dimensional cone that is dual to the 2-dimensional cone generated by P when we translate v to the origin.

Theorem 1.8 (Claim 1 on pg. 38 of [1]). *Let $P \subset \mathbb{R}^2$ be a lattice polygon with inner normal fan Δ . For a face Q of P , let $\sigma_Q \in \Delta$ be the cone corresponding Q . If \mathbf{v} is in the relative interior of σ_Q , then*

$$\lim_{z \rightarrow 0} \phi_P(\lambda_{\mathbf{v}}(z))$$

is the distinguished orbit representative of the orbit corresponding to Q .

Proof. See [1]. □

1.3. Fans and gluing varieties. Suppose that we have a fan Δ in N . For each cone $\sigma \in \Delta$ we have a ring A_σ that is the algebra on monomials with exponent vectors in $\hat{\sigma}$.

If cones σ_1 and σ_2 intersect in a face τ , then $A_{\sigma_i} \hookrightarrow A_\tau$ for each i . These ring maps correspond to $U_\tau \subset U_{\sigma_i}$ as an open subset. So, we see that the cones in N correspond to affine varieties which glue together over open sets corresponding to faces of cones in N .

Example 1.9 (\mathbb{P}^2). The three cones $\sigma_0, \sigma_1, \sigma_2$ correspond to affine varieties U_{σ_i} which correspond to the standard affine open cover of \mathbb{P}^2 by coordinate patches U_i . Let's see this and see how the ring maps give rise to the usual patching maps.

We see that $A_{\sigma_0} = \mathbb{C}[x, y], A_{\sigma_1} = \mathbb{C}[\frac{1}{x}, \frac{y}{x}]$, and $A_{\sigma_2} = \mathbb{C}[\frac{x}{y}, \frac{1}{y}]$. Each of these rings is the affine coordinate ring of a copy of \mathbb{C}^2 as they each require two generators that do not satisfy any relations.

The intersection of σ_0 and σ_1 is the face τ equal to the nonnegative y -axis. Therefore, the dual of τ is the upper halfplane and $A_\tau = \mathbb{C}[x, \frac{1}{x}, y]$. The maximal ideals of A_τ have the form $\langle x - a, y - b \rangle$. Notice however, that such an ideal contains $1 - \frac{a}{x}$. Therefore, if $\langle x - a, y - b \rangle$ is a maximal

ideal, $a \neq 0$. (Otherwise, it contains 1 so is equal to the whole ring.) i.e., $U_\tau \cong \mathbb{C}^* \times \mathbb{C}$.

We know that $A_{\sigma_0} \hookrightarrow A_\tau$. In the map of varieties from $U_\tau \rightarrow U_{\sigma_0}$ the point $(a, b) \in U_\tau$ goes to the point $(a, b) \in U_{\sigma_0}$ since the inverse image of $\langle x - a, y - b \rangle \subset A_\tau$ is just the ideal $\langle x - a, y - b \rangle \subset A_{\sigma_0}$.

We also have $A_{\sigma_1} \hookrightarrow A_\tau$. To see how to pull back the ideal $\langle x - a, y - b \rangle$ we do a little computation.

$$\langle x - a, y - b \rangle = \left\langle 1 - \frac{a}{x}, \frac{y}{x} - \frac{b}{x} \right\rangle = \left\langle \frac{1}{a} - \frac{1}{x}, \frac{y}{x} - \frac{b}{a} \right\rangle$$

This shows us that $(a, b) \in U_\tau$ is the image of $(\frac{1}{a}, \frac{b}{a}) \in U_{\sigma_1}$. (Or, that $(c, d) \in U_{\sigma_1}$ maps to $(\frac{1}{c}, \frac{d}{c}) \in U_\tau$.) We see that the points (a, b) in U_{σ_0} with $a \neq 0$ glue to the points $(\frac{1}{a}, \frac{b}{a}) \in U_{\sigma_1}$.

We'll check the other transitions in the exercises.

1.4. Refining a fan.

Example 1.10 (\mathbb{P}^2 vs X_{P_3}). Consider the inner normal fans of P_1 and P_3 . Notice that the only difference is that one of the cones has been split in two.

Definition 1.11. We say that a fan Δ' is a *refinement* of Δ if every cone in Δ is the union of cones in Δ' .

The important point is that if $\sigma' \subset \sigma$, then $A_\sigma \hookrightarrow A_{\sigma'}$. Therefore, we get a map of varieties going the other direction. More precisely,

Proposition 1.12. *If Δ' is a refinement of Δ , then there is birational morphism $X_{\Delta'} \rightarrow X_\Delta$.*

This is discussed in [1]. See the exercise and hint on pg. 18 and more on pgs. 22-23.

Example 1.13 (The blowup of \mathbb{C}^2 at $\mathbf{0}$). Consider the fan Δ consisting of cone $\{(1, 0), (0, 1)\}$ and all of its faces. Let Δ' be the fan gotten by subdividing the maximal cone of Δ by the nonnegative span of $(1, 1)$. So, Δ' is a refinement of Δ .

Let σ be the maximal cone of Δ so that $A_\sigma = \mathbb{C}[x, y]$. Let $\sigma_0 = \text{cone}\{(0, 1), (1, 1)\}$ and $\sigma_1 = \text{cone}\{(1, 1), (1, 0)\}$. We see that $A_{\sigma_0} = \mathbb{C}[x, \frac{y}{x}]$ and $A_{\sigma_1} = \mathbb{C}[\frac{x}{y}, y]$. Each of these rings is the coordinate ring of a copy of \mathbb{C}^2 . Moreover, since $A_\sigma \hookrightarrow A_{\sigma_i}$ we get maps of affine varieties $U_{\sigma_i} \rightarrow U_\sigma$.

Let's see what these maps do. The point $(a, b) \in U_{\sigma_0}$ corresponds to maximal ideal $\langle x - a, \frac{y}{x} - b \rangle$. Since

$$\langle x - a, \frac{y}{x} - b \rangle = \langle x - a, y - bx \rangle = \langle x - a, y - ba \rangle$$

we see that $(a, b) \mapsto (a, ab)$ and that the points $(0, b)$ all map to the origin.

Similarly, a point $(c, d) \in U_{\sigma_1}$ corresponds to the ideal $\langle \frac{x}{y} - a, y - b \rangle$ which pulls back to the ideal $\langle x - ab, y - b \rangle$ in A_σ . Therefore, $(c, d) \mapsto (cd, d)$ and all points of the form $(c, 0)$ go to the origin.

In fact, the U_{σ_i} are exactly the standard affine patches covering the blowup of \mathbb{C}^2 at the origin. To see this, let's glue them together. The cones intersect in the ray τ spanned by $(1, 1)$ so $A_\tau = \mathbb{C}[x, \frac{x}{y}, \frac{y}{x}]$.

Let's see which points $(a, b) \in U_{\sigma_0}$ gets glued to points of U_{σ_1} . The ideal $\langle x - a, \frac{y}{x} - b \rangle \subset A_{\sigma_0}$ is the inverse image of an ideal with the same generators in A_τ . The ideal in A_τ can be rewritten so that we can tell what its inverse image in A_{σ_1} is as follows. We want to see an ideal of the form $\langle \frac{x}{y} - c, y - d \rangle \subset A_{\sigma_1}$.

Since

$$-\frac{1}{b} \frac{x}{y} (\frac{y}{x} - b) = \frac{x}{y} - \frac{1}{b}$$

and

$$\frac{y}{x} (x - a) + a (\frac{y}{x} - b) = y - ab$$

we see that $(a, b) \in U_{\sigma_0}$ corresponds to $(\frac{1}{b}, ab) \in U_{\sigma_1}$ and that for this correspondence to hold, $b \neq 0$.

Here is a chart that summarizes the information of which points from U_{σ_0} and U_{σ_1} get glued together and where they go in U_σ under the "blow down" map.

U_{σ_0}	U_{σ_1}	U_σ	conditions for gluing
(a, b)	$(\frac{1}{b}, ab)$	(a, ab)	$b \neq 0$
$(a, 0)$	—	$(a, 0)$	—
$(0, b)$	$(\frac{1}{b}, 0)$	$(0, 0)$	$b \neq 0$
$(cd, \frac{1}{d})$	(c, d)	(cd, d)	$d \neq 0$
—	$(c, 0)$	$(0, 0)$	—
$(0, \frac{1}{d})$	$(0, d)$	$(0, d)$	$d \neq 0$

1.5. Toric resolution of singularities. We have seen that a toric surface is nonsingular at a fixed point corresponding to vertex v if and only if the first lattice vectors along the cone edges meeting at v formed a \mathbb{Z} -basis for \mathbb{Z}^2 .

Question 1.14. Can we resolve these singularities staying in the toric world? i.e., can we find a smooth toric surface that maps birationally to a given singular toric variety?

The answer is – yes! Given any fan, we can subdivide cones until each cone corresponds to a smooth affine toric patch. We get a toric

resolution of singularities. We'll work out examples of toric resolution of singularities of surfaces tomorrow.

2. EXERCISES

- (1) Show that if $\tau \subset \sigma$, then $\hat{\sigma} \subset \hat{\tau}$.
- (2) Compute all of the transition functions for the standard open affine patches of \mathbb{P}^2 using ring maps as in Example 1.9.
- (3) Translate our coordinate systems for the blowup of \mathbb{C}^2 at the origin into the coordinates of $\mathbb{C}^2 \times \mathbb{P}^1$ given in week 2.
- (4) Can you use the ideas from today's lecture to describe how X_{P_3} is related to X_{P_4} ? It might be helpful to compare their inner normal fans.

REFERENCES

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