

TORIC SURFACES
PCMI UNDERGRADUATE COURSE, JULY 2008

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1. DAY 5

After today's lecture you should be able to

- (1) resolve the singularities of a toric surface algorithmically.
- (2) understand the (statement of the) classification of smooth toric surfaces.

Remark 1.1. Recall our basic setup. We have a lattice polytope $P \subset M_{\mathbb{R}} \cong \mathbb{R}^d$ with $P \cap M = \{\mathbf{a}_0 = \mathbf{0}, \dots, \mathbf{a}_n\}$. Then if $A = (\mathbf{a}_1 \cdots \mathbf{a}_n)$, the affine variety $X_A = X_P \cap U_0$ has coordinate ring $\mathbb{C}[\mathbf{t}^{\mathbf{a}_1}, \dots, \mathbf{t}^{\mathbf{a}_n}] \cong \mathbb{C}[x_1, \dots, x_n]$.

Note: Let σ be the dual of the cone generated by P . For the ring $\mathbb{C}[\mathbf{t}^{\mathbf{a}_1}, \dots, \mathbf{t}^{\mathbf{a}_n}]$ to be the algebra spanned by monomials whose exponent vectors are the elements of $\hat{\sigma} \cap M$, we must be able to write every element of $\hat{\sigma} \cap M$ as a sum of elements from the set $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$. This is the same as saying that the lattice points in P must generate $\hat{\sigma} \cap M$ as a semigroup.

We need the condition above to hold for P at every vertex in order to say that X_P is isomorphic to the abstract toric variety that we associate to inner normal fan Δ of P . In more advanced language, we need P to correspond to a very ample divisor. This is always true if P is a polygon or if P has an inner normal fan corresponding to a smooth variety. However, in dimensions greater than 2, P may correspond to a divisor that is merely ample on the abstract variety associated to its inner normal fan. (See pgs. 70-72 of [1] for a discussion.)

The upshot of this remark is that everything we've done works in the 2-dimensional case, and in higher dimensions more care is needed.

In today's lecture we are following sections 2.2, 2.5, and 2.6 of [1].

Lemma 1.2. *Let σ be a 2-dimensional cone and $\mathbf{v}_1, \mathbf{v}_2$ be the first lattice vectors along its 1-dimensional faces (listed counter clockwise).*

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Then there exists an invertible 2×2 integer matrix B so that $B\mathbf{v}_1 = (m, -k)$ where $0 \leq k < m$ are integers and $B\mathbf{v}_2 = (0, l)$.

Proof. This is an exercise. \square

Example 1.3. Consider the cone below where $\mathbf{v}_1 = (2, -1)$ and $\mathbf{v}_2 = (0, 1)$. Since $\det \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} = 2$, the vectors are not a \mathbb{Z} -basis for \mathbb{Z}^2 . Therefore, the origin is a singular point of the corresponding affine variety.

Subdivide this cone by adding in the ray spanned by $(1, 0)$. Taking since $\det \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} = 1$ and $\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$, the affine varieties associated to the new 2-dimensional cones are both smooth!

From yesterday, we know that adding in this new ray exactly corresponds to blowing up the origin in our original affine variety. In this case, we get a smooth variety after just one blowup.

In some cases, we may need to repeat this subdivision procedure multiple times, but we can give an algorithm (a set of steps which terminates) to resolve any singularity.

Algorithm 1.4. INPUT: A cone σ generated by $(m_0, -k_0)$, with $0 \leq -k_0 < m_0$ and $(0, 1)$.

OUTPUT: A fan Δ corresponding to a smooth toric variety X_Δ that is gotten by starting with U_σ and then blowing up finitely many times.

Let Δ be the fan consisting of σ and all of its faces.

- (1) Set Δ equal to the subdivision of the current fan Δ gotten by adding in the ray spanned by $(1, 0)$.
- (2) Let $\Delta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Delta$. (Rotate Δ by 90 degrees.)
- (3) Let $\Delta = \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix} \Delta$ for some value of a that takes (k_i, m_i) to $(m_{i+1}, -k_{i+1})$ where $m_{i+1} = k_i$ and $0 \leq k_{i+1} < m_{i+1}$. (This *shears* the plane.)
- (4) If $k_{i+1} = 0$, then stop. Otherwise, let $i = i + 1$ and repeat.

Example 1.5. Let σ be the cone generated by $(3, -2)$ and $(0, 1)$. After we add the ray generated by $(1, 0)$, the cone in the first quadrant is smooth, but the one below the first quadrant is not. After the rotation in step (2) of the algorithm, we shear by $\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$.

The nonsmooth cone is now generated by $(2, -1)$ and $(0, 1)$. Applying the algorithm again, we insert the ray spanned by $(1, 0)$. We now check that each cone is generated by a \mathbb{Z} -basis for \mathbb{Z}^2 .

Theorem 1.6. *If X_Δ is a smooth projective toric surface, then it can be constructed by blowing up finitely many times starting with either \mathbb{P}^2 or a Hirzebruch surface corresponding to the fan with rays $(1, 0), (0, 1), (-1, t), (0, -1)$.*

Proof. A proof is outlined in the exercises in section 2.5 of [1]. \square

REFERENCES

- [1] William Fulton, *Introduction to Toric Varieties*, Annals of Math. studies no. 131, Princeton Univ. Press, Princeton, 1993.
- [2] Israel M. Gelfand, Mikhail M. Kapranov, and Andrei V. Zelvinsky, *Discriminants, Resultants and Multidimensional Determinants*, Birkhäuser, Boston, 1994.
- [3] Bernd Sturmfels, *Gröbner Bases and Convex Polytopes*, Univ. Lecture Series, v. 8, Amer. Math. Soc. , Providence, 1996.

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