Secants of Edge Ideals and Their Betti Diagrams

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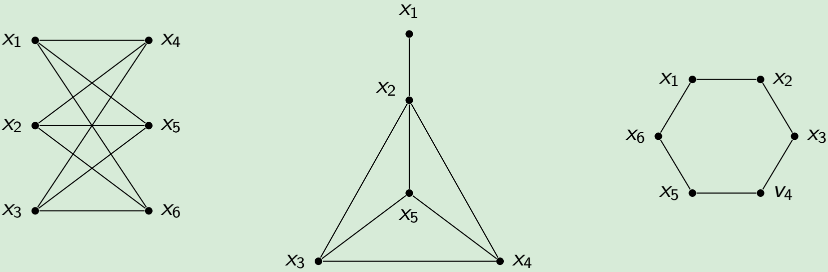
What is a graph?

Definition

A *graph* G is an ordered pair (V, E) where V is a set of vertices and the edges E are a set of two-element subsets of V.

Graphs are best thought of visually.

Example

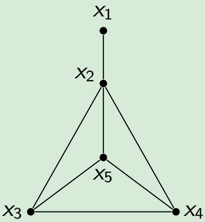


The Edge Ideal of a Graph

Let $S = \mathbb{R}[x_1, \ldots, x_n]$ be the set of polynomials in n variables whose coefficients are real numbers.

Example

If $G =$



then the *edge ideal* of G is

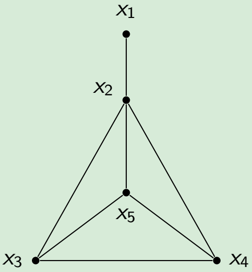
$$\begin{aligned}\mathcal{I}(G) &= \langle x_1x_2, x_2x_3, x_2x_4, x_2x_5, x_3x_4, x_3x_5, x_4x_5 \rangle \\ &= \{c_0x_1x_2 + c_1x_2x_3 + c_2x_2x_4 + c_3x_2x_5 \\ &\quad + c_4x_3x_4 + c_5x_3x_5 + c_6x_4x_5 \mid c_0, \ldots, c_6 \in S\}\end{aligned}$$

Secant Ideals

For an edge ideal, it has been shown [?] that its secant ideal is generated by products of variables whose corresponding vertices form cycles of three.

Example

The graph G



has secant ideal

$$\mathcal{I}(G)^{\{2\}} = \langle x_1x_2x_4, x_1x_3x_4, x_2x_3x_4 \rangle .$$

Free Resolutions and Syzygies

Definition

Let $I = \langle f_0, \ldots, f_r \rangle$ be an edge ideal in $S = \mathbb{R}[x_1, \ldots, x_n]$. A *free resolution* of I is a sequence of maps ϕ_i

$$\cdots \xrightarrow{\phi_2} \begin{matrix} S \\ \oplus \\ \vdots \\ \oplus \\ S \end{matrix} \xrightarrow{\phi_1 = [\cdots]} \begin{matrix} S \\ \oplus \\ \vdots \\ \oplus \\ S \end{matrix} \xrightarrow{\phi_0 = [f_1 \cdots f_r]} S$$

such that $\operatorname{im} \phi_{i+1} = \ker \phi_i$.

Definition

A *syzygy* is an element of $\ker \phi_i$ for some i.

Betti Diagrams

Information about free resolutions and syzygies can be stored in a betti diagram.

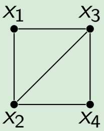
A betti diagram:

- column $i \leftrightarrow$ dimension of $\ker \phi_i$
- keeps track of the degree of the monomials in syzygies

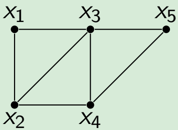
"Zigzags": A Special Kind of Triangulated Graph

Example

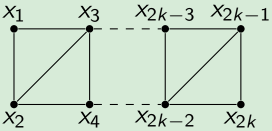
Zigzag Graph G_4



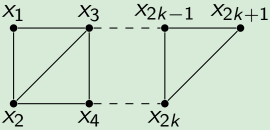
Zigzag Graph G_5



Zigzag Graph G_{2k}



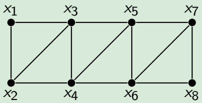
Zigzag Graph G_{2k+1}



The Betti Diagrams of "Zigzag" Secants

Let I_n be the edge ideal of zigzag graph G_n.

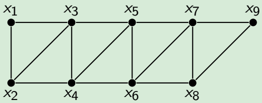
Example



$$I_8^{\{2\}} = \langle x_1x_2x_3, x_2x_3x_4, x_3x_4x_5, x_4x_5x_6, x_5x_6x_7, x_6x_7x_8 \rangle$$

betti diagram for $I_8^{\{2\}}$:

	0	1	2	3	4
0	1	.	.	.	.
1	.	.	.	.	.
2		6	5	.	.
3	.	.	.	.	.
4	.	.	3	4	1



$$I_9^{\{2\}} = \langle x_1x_2x_3, x_2x_3x_4, x_3x_4x_5, x_4x_5x_6, x_5x_6x_7, x_6x_7x_8, x_7x_8x_9 \rangle$$

betti diagram for $I_9^{\{2\}}$:

	0	1	2	3	4
0	1	.	.	.	.
1	.	.	.	.	.
2	.	7	6	.	.
3	.	.	.	.	.
4	.	.	6	9	3

The Betti Diagrams of "Zigzag" Secants

Theorem

Conjecture

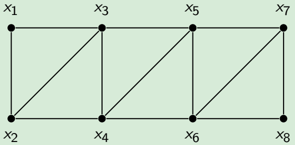
For a zigzag graph G_n, the secant of the edge ideal has a betti diagram of the form

	0	1	2	3	4
0	1	.	.	.	.
1					

Linear Syzygies on Zigzags: An Example

Question: What are the linear syzygies in $\ker \phi_0$?

Example



	0	1	2	3	4
0	1	.	.	.	.
1	.	.	.	.	.
2		6	5	.	.
3	.	.	.	.	.
4	.	.	3	4	1

	0	1	2
0	1	.	.
1	.	.	.
2	.	$n-2$	$n-3$

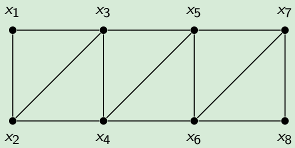
- Linear syzygies in $\ker \phi_0$ correspond to three-cycles that share an edge

$$x_4(x_1x_2x_3) - x_1(x_2x_3x_4) = 0$$
$$x_5(x_2x_3x_4) - x_2(x_3x_4x_5) = 0$$
$$x_6(x_3x_4x_5) - x_3(x_4x_5x_6) = 0$$
$$x_7(x_4x_5x_6) - x_4(x_5x_6x_7) = 0$$
$$x_8(x_5x_6x_7) - x_5(x_6x_7x_8) = 0$$

Quadratic Syzygies on Zigzags

Quadratic syzygies in $\ker \phi_0$ correspond to three-cycles that are exactly one three-cycle "apart," i.e., that have exactly one vertex in common.

Example



$$x_4x_5(x_1x_2x_3) - x_1x_2(x_3x_4x_5) = 0$$

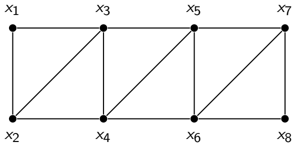
However, all of these syzygies can be written as a linear combination of two linear syzygies:

$$\begin{aligned} & x_4x_5(x_1x_2x_3) - x_1x_2(x_3x_4x_5) \\ =& x_4x_5(x_1x_2x_3) - x_1x_2(x_3x_4x_5) \\ & + (x_1x_2x_3x_4x_5 - x_1x_2x_3x_4x_5) \\ =& x_5(x_4(x_1x_2x_3) - x_1(x_2x_3x_4)) \\ & + x_1(x_5(x_2x_3x_4) - x_2(x_3x_4x_5)), \end{aligned}$$

so they are not necessary to minimally generate the kernel.

Cubic Syzygies: Case One

We expect cubic syzygies when two three-cycles are at least two three-cycles "apart" in the zigzag.

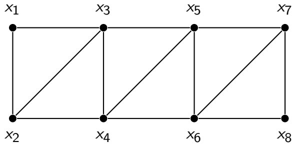


When the three-cycles are exactly two three-cycles apart:

$$
\begin{aligned}
& x_4x_5x_6(x_1x_2x_3) - x_1x_2x_3(x_4x_5x_6) \\
= & x_5x_6(x_4(x_1x_2x_3) - x_1(x_2x_3x_4)) \\
& + x_1x_6(x_5(x_2x_3x_4) - x_1(x_3x_4x_5)) \\
& + x_1x_2(x_6(x_3x_4x_5) - x_3(x_4x_5x_6),
\end{aligned}
$$

so again this syzygy is a linear combination of linear syzygies.

Cubic Syzygies: Case 2



If the three-cycles are more than two three-cycles apart, the associated syzygy cannot be written as a linear combination of the linear syzygies and/or the other cubic syzygies.

So there are

$$1 + 2 + \cdots + (n-6) = \frac{1}{2}(n-6)(n-5)$$

cubic syzygies that minimally generate the kernel.

	0	1	2	3	4
0	1	.	.	.	.
1	.	.	.	.	.
2	.	$n-2$	$n-3$	.	.
3	.	.	.	.	.
4	.	.	$\frac{1}{2}(n-6)(n-5)$		

Acknowledgments & References

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References