# EDGE AND SECANT IDEALS OF SHARED-VERTEX GRAPHS

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ABSTRACT. We examine minimal free resolutions and Betti diagrams of the edge and secant ideals of one family of graphs. We demonstrate how splitting the graph explains patterns observed in the Betti diagrams.

## 1. Introduction

We will focus on a family of graphs that we will call *shared-vertex graphs*, which consist of distinct triangles joined together at one vertex.

Each graph corresponds to a polynomial ring containing variables corresponding to each vertex. When we discuss the *edge ideals* of a graph, we are referring to the ideal generated by the degree-two monomials that correspond to the edges of the graph. For instance, the edge $AB$ corresponds to the monomial $ab$.

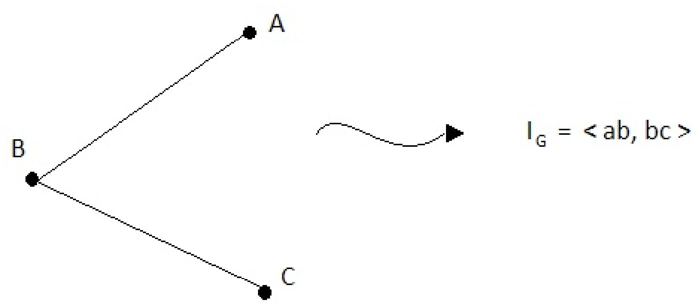


FIGURE 1. A Graph and Its Corresponding Edge Ideal

The *minimal free resolution* of a given edge ideal, will refer to the exact sequence defined by its final homomorphism: multiplication by the generators of our ideal. The *Betti diagram* summarizes the number of inputs and degree at each step of the resolution.

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Use of the Macaulay 2 program has demonstrated a predictable pattern in the Betti diagrams of the edge and secant ideals of the shared-vertex graphs. In §2 we demonstrate the pattern in the Betti Diagrams. In §3 we demonstrate a convenient splitting for our family of graphs. In §4 we show how the splitting explains the pattern in the Betti diagrams.

## 2. Betti Diagrams

After examining the cases starting with two triangles joined at a vertex, and going up to 6 triangles joined at a central vertex, the Betti Diagrams follow a very predictable pattern. Let $n$ be the number of triangles in the graph. Then the Betti Diagram for the edge ideal will be of the following form:

| | 0 | 1 | 2 | 3 | ... | $n$ | $n+1$ | ... | $2n$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | | | |
| 2 | | $\binom{2n}{1}+\binom{n}{1}$ | $\binom{2n}{2}+\binom{n}{1}$ | $\binom{2n}{3}$ | ... | $\binom{2n}{n}$ | $\binom{2n}{n+1}$ | ... | $\binom{2n}{2n}$ |
| 3 | | | $\binom{n}{2}$ | $\binom{n}{2}$ | | | | | |
| 4 | | | | $\binom{n}{3}$ | ... | | | | |
| ... | | | | | ... | $\binom{n}{n-1}$ | | | |
| n | | | | | | $\binom{n}{n}$ | $\binom{n}{n}$ | | |

Looking at this Betti diagram, three patterns are obvious:

- The top row, which gives the binomial coefficients of $2n$.
- The two diagonals, containing the binomial coefficients of $n$.

There is overlap among these three patterns in the first two columns of the top row.

As for the secant ideal, the Betti Diagram is even simpler:

| | 0 | 1 | 2 | ... | $n$ |
|---|---|---|---|---|---|
| 1 | 1 | | | | |
| 2 | | | | | |
| 3 | | $\binom{n}{1}$ | | | |
| 4 | | | $\binom{n}{2}$ | | |
| ... | | | | ... | |
| n | | | | | $\binom{n}{n}$ |

Here we only have one diagonal pattern of the binomial coefficients up to $n$.

In the remaining sections, we will show that these patterns will persist for all values of $n$.

## 3. Splitting the Graph

To try and make sense of these patterns, let's look again at one example of a shared-vertex graph. Below is the graph with $n = 3$ triangles.

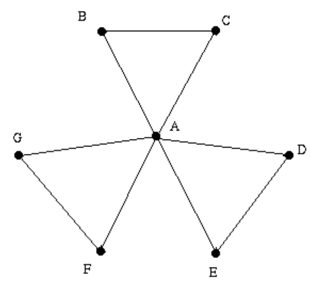


Figure 2. 3-Triangle Graph

You may notice that you can divide the edges into two categories:

(1) The $2n$ edges aligned radially from the center, which we will call $r$-edges.
(2) The $n$ edges aligned along the perimeter, which we will call $p$-edges.

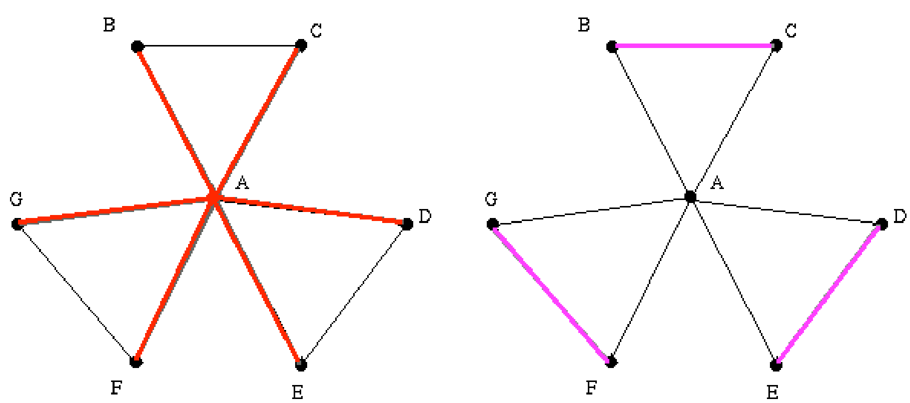


Figure 3. Highlighted $r$-edges and $p$-edges

This distinction lends itself to a convenient splitting on our graph, a technique described in [1], and paraphrased below.

**Definition 3.1.** (Eliahou-Kervaire) A monomial ideal $I$ is splittable if $I$ is the sum of two nonzero monomial ideals $J$ and $K$, that is, $I = J + K$, such that

(1) The minimal generating set $\mathcal{G}(I)$ is the disjoint union of $\mathcal{G}(J)$ and $\mathcal{G}(K)$, and
(2) there is a **splitting function**

$$\mathcal{G}(J \cap K) \longrightarrow \mathcal{G}(J) \times \mathcal{G}(K)$$

$$w \longmapsto (\phi(w), \psi(w))$$

such that:
(1) for all $w \in \mathcal{G}(J \cap K)$, $w = lcm(\phi(w), \psi(w))$.
(2) for every subset $\mathcal{S} \subset \mathcal{G}(J \cap K)$, both $lcm(\phi(\mathcal{S}))$ and $lcm(\psi(\mathcal{S}))$ strictly divide $lcm(\mathcal{S})$

Given this definition, we will split up our shared vertex graphs into the set of $r$-edges and the set of $p$-edges, as is shown in the images below.

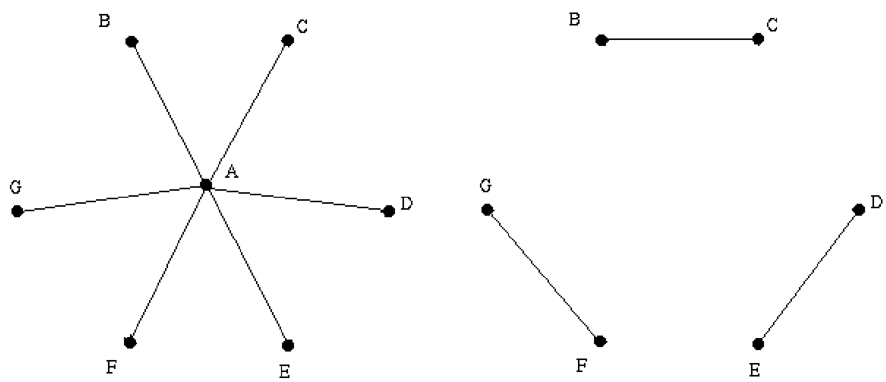


FIGURE 4. The subgraphs corresponding to J and K

Switching now to the polynomial ring $k[x_0, ..., x_{2n}]$ for convenience, we can describe the edge ideal in the $n$ triangle case, and its component ideals as follows:

(1) $I =< x_0x_1, x_0x_2, x_1x_2, ..., x_0x_{2n-1}, x_0x_{2n}, x_{2n-1}x_{2n} >$, the entire edge ideal.
(2) $J =< x_0x_1, x_0x_2, ..., x_0x_{2n-1}, x_0x_{2n} >= x_0 < x_1, x_2, ..., x_2n >$, the ideal generated by the $r$-edges.
(3) $K =< x_1x_2, x_3x_4, ..., x_{2n-1}x_{2n} >$, the ideal generated by the $p$-edges.
(4) $J \cap K =< x_0x_1x_2, x_0x_3x_4, ..., x_0x_{2n-1}x_{2n} >$, the intersection of the ideals (which incidentally coincides with our secant ideal), or the product of the variables representing the vertices of each triangle.

We can define the splitting function to take one of the degree-3 monomials in the intersection to two degree-2 monomials :

$$\mathcal{G}(J \cap K) \longrightarrow \mathcal{G}(J) \times \mathcal{G}(K)$$

$$w \longmapsto \left( \frac{w}{\gcd(w, x_2 x_4 ... x_{2n})}, \frac{w}{x_0} \right)$$

For instance, take any triangle in the intersection $J \cap K$, which will have general form $x_0 x_{2i-1} x_{2i}$. The $\phi$ function will get rid of $x_{2i}$, leaving $x_0 x_{2i-1}$, an $r$-edge; the $\psi$ function gets rid of the $x_0$, leaving $x_{2i-1} x_{2i}$, a $p$-edge.

The required properties for a splitting function are easily seen to be satisfied. (Note that any similar product of unconnected vertices would work for the gcd in the $\phi$ function.)

## 4. Components of the Betti Diagram

Having established that our edge ideal is splittable, we can utilize a powerful relation about graded Betti numbers established by Fattabi, also reported in [1].

**Theorem 4.1.** *Suppose $I$ is a splittable monomial ideal with splitting $I = J + K$. Then*

$$\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K) \text{ for all } i, j \geq 0$$

$$\text{where } \beta_{i-1,j}(J \cap K) = 0 \text{ if } i = 0.$$

Now, we have to look only at the much simpler ideals $J$, $K$, and $J \cap K$, in order to describe the Betti diagram of $I$.

First, let's look at $J$. In the $n$ triangle case, this ideal is generated by $2n$ quadratic monomials, each containing a factor of $x_0$ and another $x_i$ unique to that monomial. Since $x_0$ is in every term, it will not appear in any of the syzygies, so we can consider this an ideal of degree-1 monomials $x_1, .., x_{2n}$. This is a Koszul complex, whose Betti diagram is of the form (as described in [2]):

| | 0 | 1 | 2 | ... | $2n$ |
|---|---|---|---|---|---|
| 1 | 1 | | | | |
| 2 | | $\binom{2n}{1}$ | $\binom{2n}{2}$ | ... | $\binom{2n}{2n}$ |

The ideal $K$ is generated by the $n$ quadratic monomials corresponding to the outer edges. None of these edges share any vertices, thus the monomials are all relatively prime. This too forms a slightly modified

Koszul complex. In this complex, each syzygy will have only degree-2 terms, since for any two monomials $m_i$ and $m_j$, $m_i$'s coefficient in the syzygy, i.e. $\frac{lcm(m_i,m_j)}{m_j}$, will simply be the other quadratic monomial. So, the Betti diagram will look the same, except each step in the resolution will be one degree higher than the previous one, as in the following diagram.

| | 0 | 1 | 2 | ... | $n$ |
|---|---|---|---|---|---|
| 1 | 1 | | | | |
| 2 | | $\binom{n}{1}$ | | | |
| 3 | | | $\binom{n}{2}$ | | |
| ... | | | | ... | |
| n | | | | | $\binom{n}{n}$ |

As we mentioned previously, the ideal $J \cap K$ is equivalent to the secant ideal, whose Betti diagram appeared earlier. We can explain this pattern by noting that, as in $J$, every term has an $x_0$ factor that can be disregarded; and as in $K$, each term has two factors unshared by the other terms. Therefore, we once again have a Koszul complex, with degree-2 monomials.

Using Kattabi's relation for the graded Betti numbers, we start by simply adding the Betti diagrams for $J$ and $K$ together. The $J \cap K$ terms are shifted over one column, because we take the $\beta_{\mathbf{i\text{-}1},j}(J \cap K)$ in the $\beta_{i,j}(I)$. Then, they are shifted up one row as well, because the rows are indexed according to the column. What we are left with is the Betti diagram shown in §1.

## References

[1] Huy Tài Hà and Adam Van Tuyl. Monomial Ideals, Edge Ideals of Hypergraphs, and their Minimal Graded Free Resolutions. *Journal of Algebraic Combinatorics: An International Journal* Volume 27, Issue 2. (April 2008) Pages 215-245

[2] David Eisenbud, with a chapter by Jessica Sidman. Lectures on the Geometry of Syzygies. *Trends in Commutative Algebra* Volume 51. (2004) Pages 115-152

[3] Grayson, Daniel R. and Stillman, Michael E., Macaulay 2, a software system for research in algebraic geometry, Available at http://www.math.uiuc.edu/Macaulay2/