The Polytopes and Toric Varieties REU group

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Mount Holyoke Summer REU 2005

Introduction

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The goal of this presentation is to understand the inner normal fans smooth lattice polytopes with lattice free edges.

Convexity

Definition: A set $X \subseteq \mathbb{R}^n$ is called *convex* if

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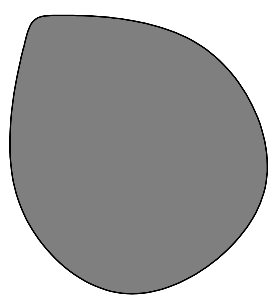
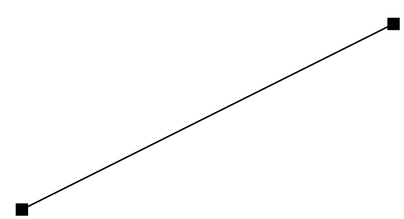


Convex



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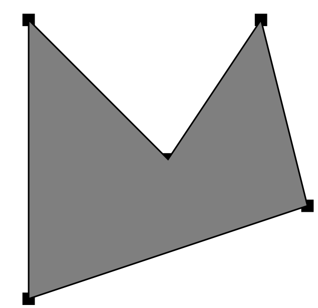

Not Convex

Convex Hulls

Definition: If $X \subseteq \mathbb{R}^n$ is any set, then the *convex hull* of X is the smallest convex set which contains X.

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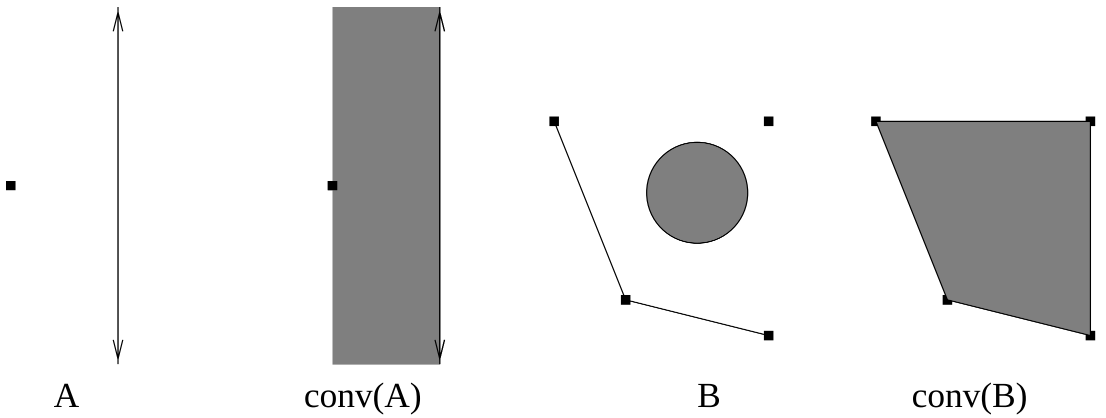

Polytopes

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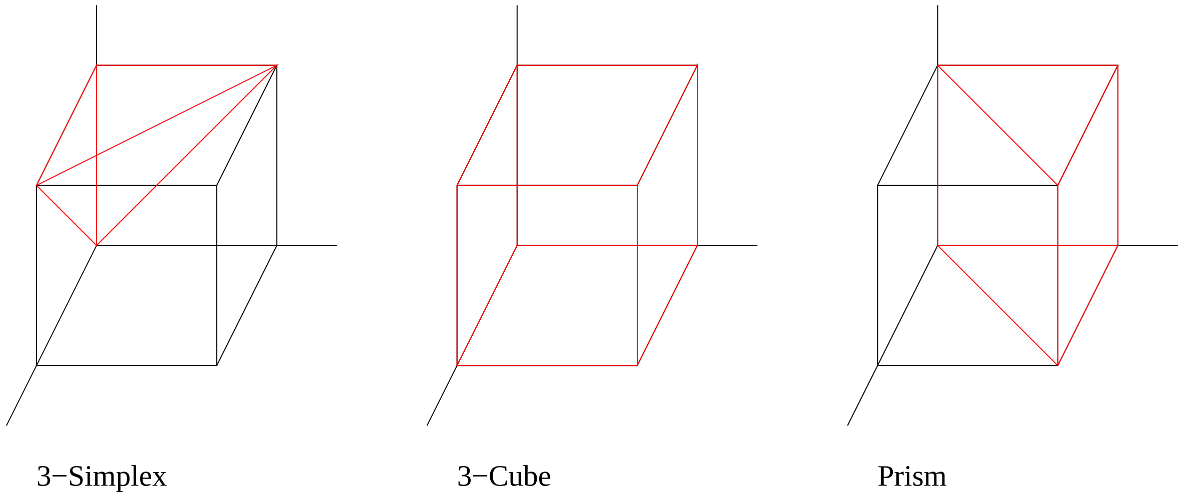
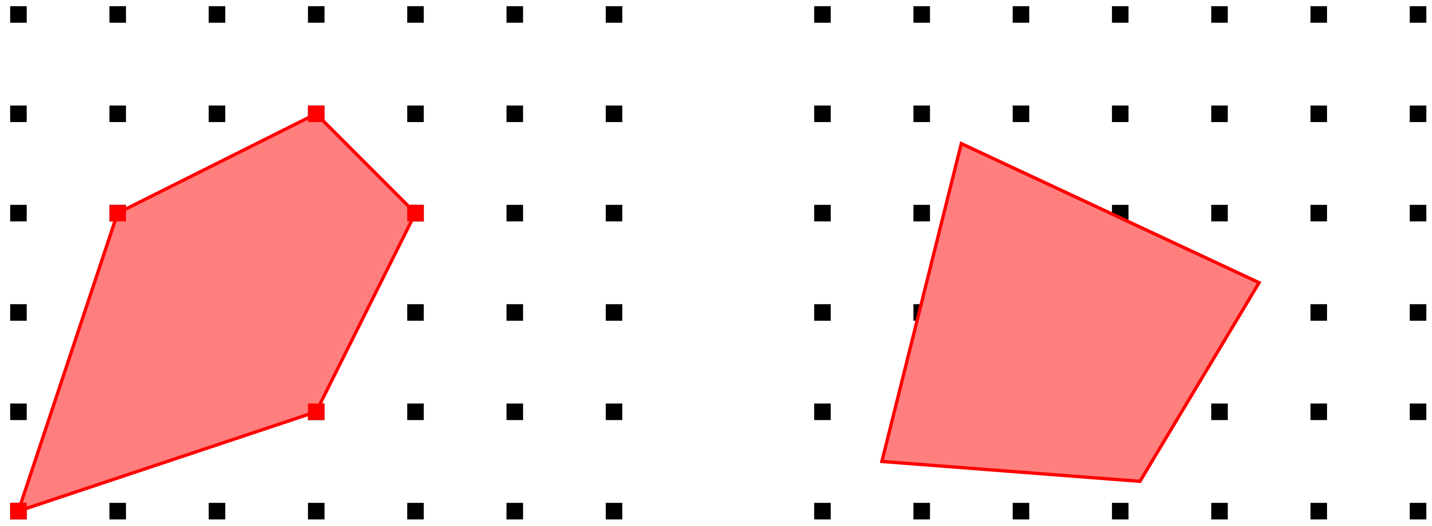


Lattice Polytopes

Definition: A *lattice polytope* is a polytope $P \subseteq \mathbb{R}^n$ with vert$(P) \subseteq \mathbb{Z}^n$.

Example:

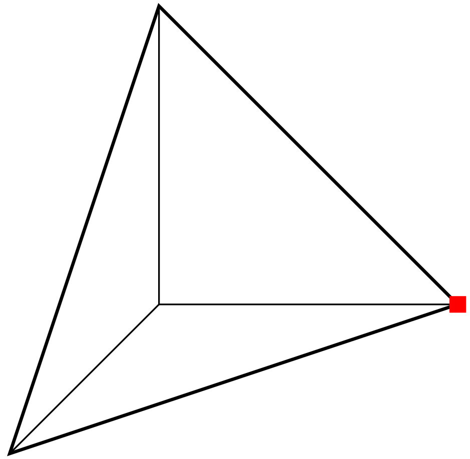


Smooth Polytopes

Definition: A polytope ($P \subseteq \mathbb{R}^n$) is *smooth* at the vertex $p \in P$ if the set

$$\{ (w_1 - p), \ldots, (w_k - p) \}$$

can be extended to a basis for $\mathbb{Z}^n$. A *smooth polytope* is a polytope which is smooth at every one of its vertices.



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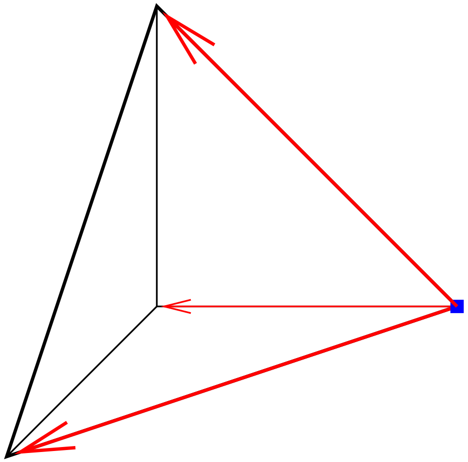

Definition: Given a set of vectors V the *cone generated by* V is

$$\text{cone}(V) = \{ \sum_{i=1}^{k} \lambda_i v_i \mid \lambda_i > 0 \ , \ v_i \in V \}$$

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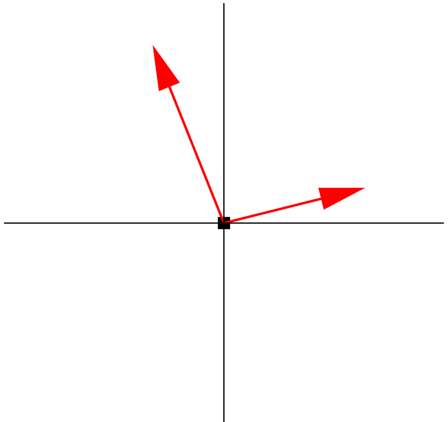

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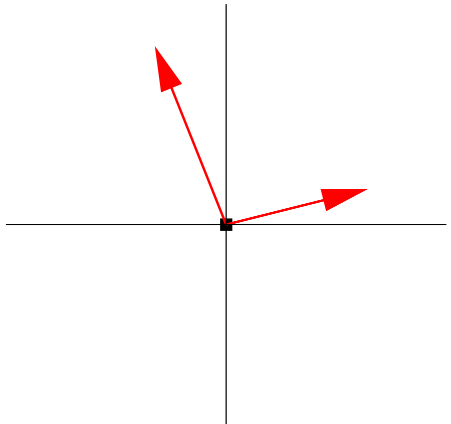

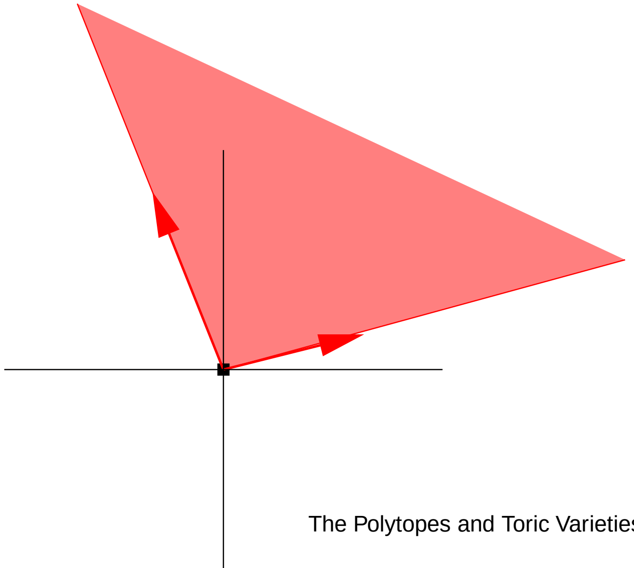

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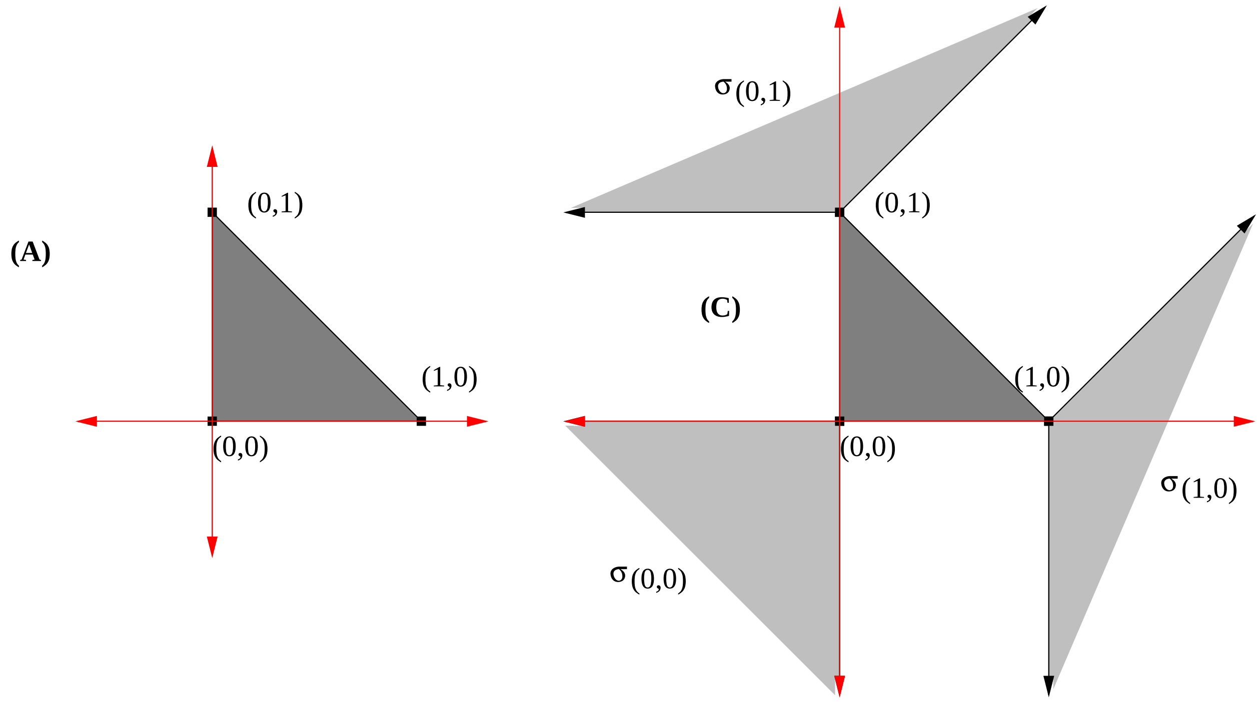

Example

$$\sigma(0,0) = \text{cone}\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \}$$

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Fans

Definition: A *fan* ($\triangle$) is a collection of cones which satisfy

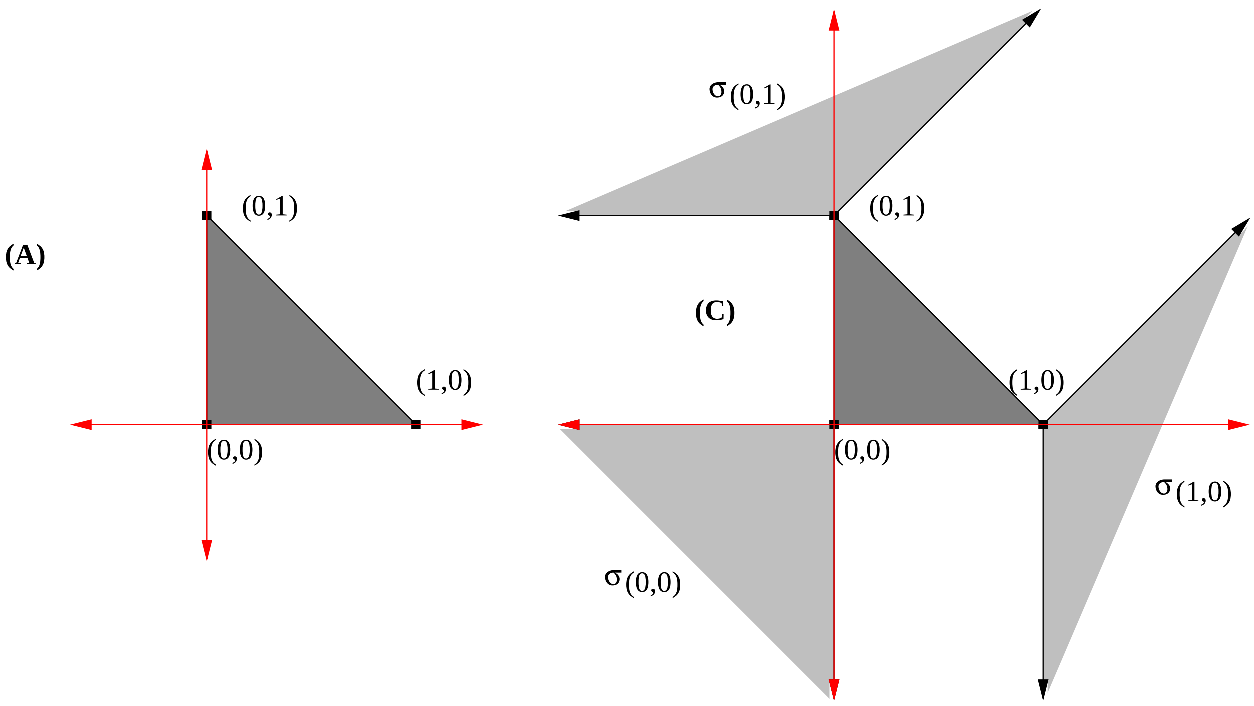
- If τ is a face of σ and $\sigma \in \triangle$, then $\tau \in \triangle$.
- If $\tau, \sigma \in \triangle$ then $\tau \cap \sigma$ is a face of both τ and σ.

Definition: If $P \subseteq \mathbb{R}^n$ is a polytope, then the associated *inner normal fan* $(\triangle_P)$ is the collection of cones

$$\triangle_P = \{ \text{cone}(F) \mid F \text{ a face of } P \}$$

Normal Fans

Example: Inner and outer normal fans



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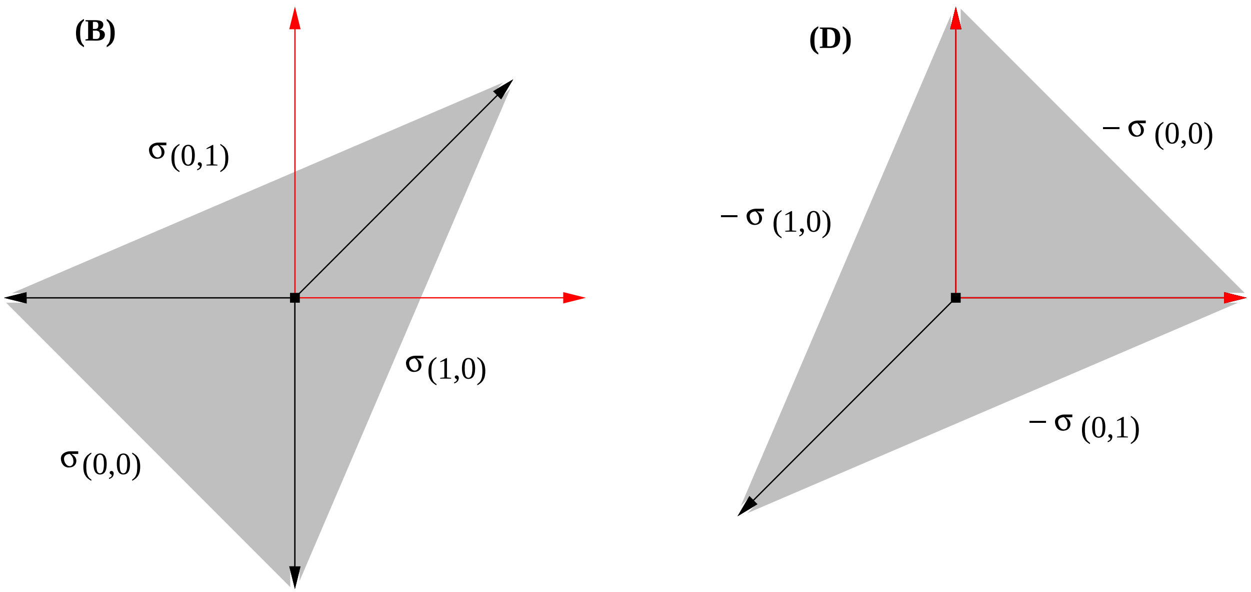
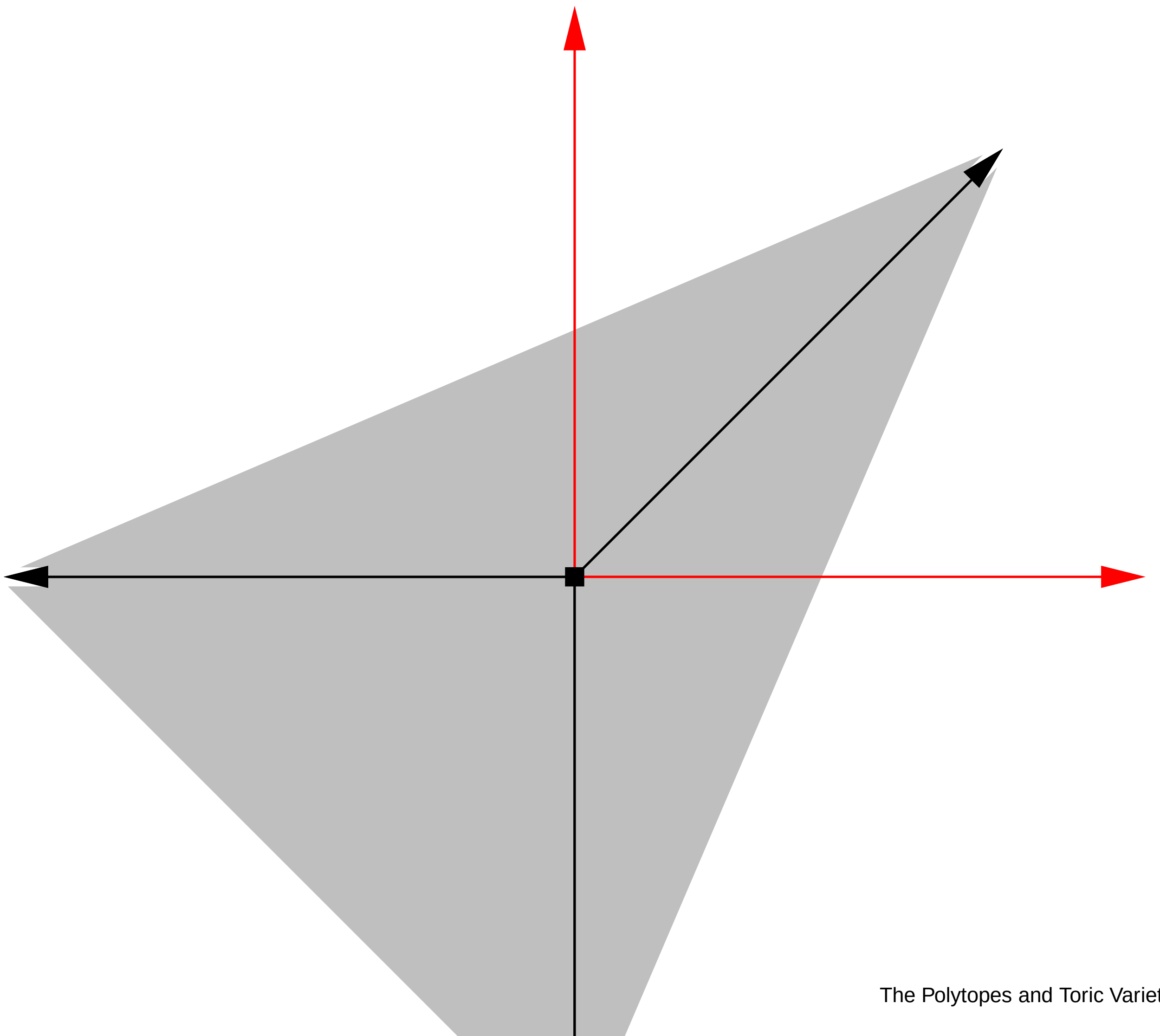


Polytopes from fans



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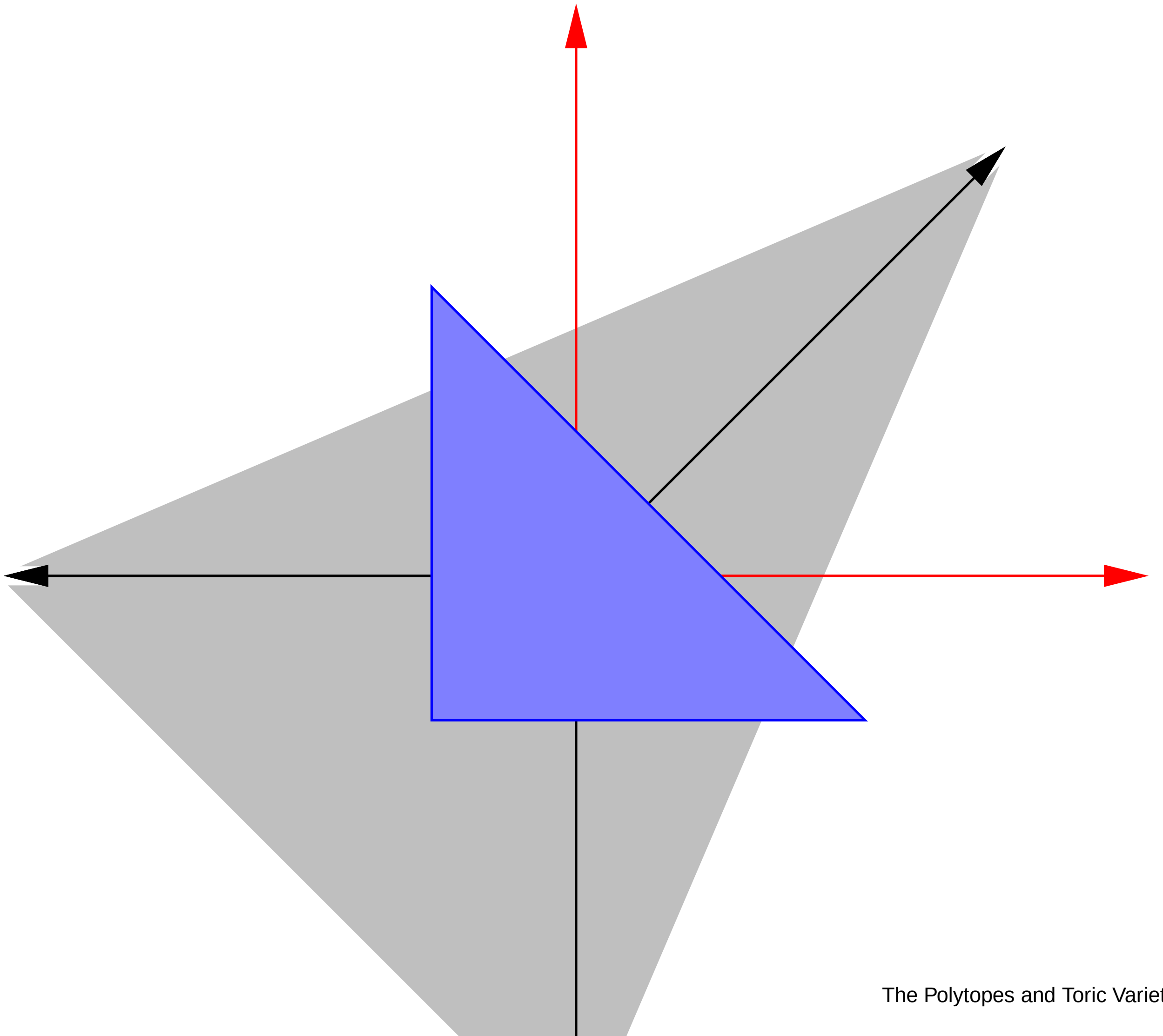

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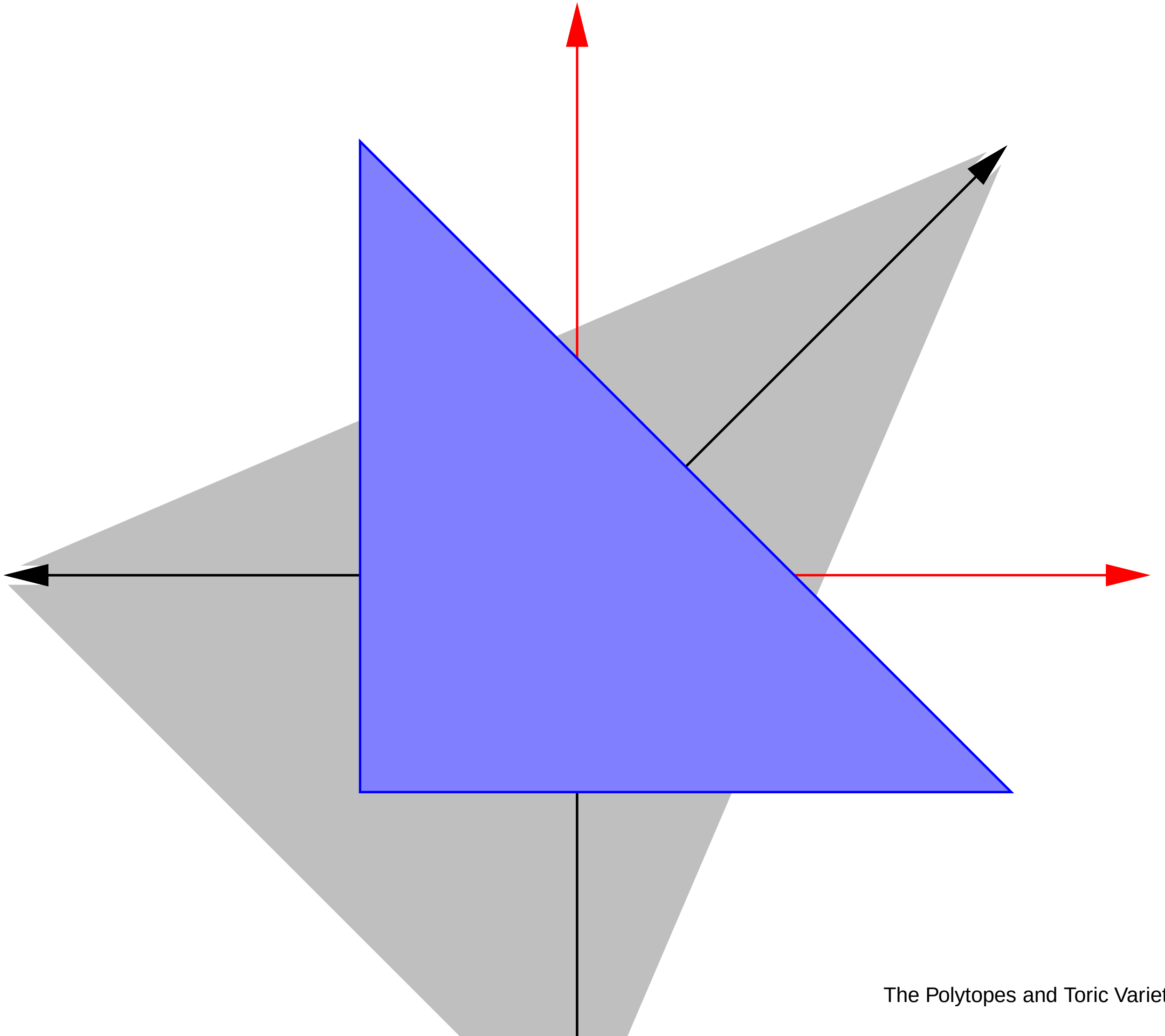

Main Theorem

Theorem 0.1 *If P, $Q \subseteq \mathbb{R}^n$ are both selfatopes and $\triangle_P = \triangle_Q$, then there exists a vector $r \in \mathbb{R}^n$ so that*

$$P = r + Q$$

Main Theorem

Theorem 0.2 *If* P*,* $Q \subseteq \mathbb{R}^n$ *are both selfatopes and* $\triangle_P = \triangle_Q$*, then there exists a vector* $r \in \mathbb{R}^n$ *so that*

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Toric Varieties

Suppose $P \subseteq \mathbb{R}^n$ is a polytope with $P \cap \mathbb{Z}^n = \{\, a_1, \ldots , a_m \,\}$. Define $\varphi_P : (\mathbb{C}^*)^n \to \mathbb{C}P^m$ to be the map

$$\varphi_P(\underline{z}) = [\underline{z}^{\underline{a_1}} : \ldots : \underline{z}^{\underline{a_m}}]$$

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Example: For the 2-simplex we have

$$\varphi_P(z_1,\, z_2) = [z_1^0 z_2^0 \,:\, z_1^1 z_2^0 \,:\, z_1^0 z_2^1] = [1 \,:\, z_1 \,:\, z_2]$$

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<u>Definition:</u> The *projective toric variety* is $X_P = \overline{\text{im}(\varphi_P)}$

Acknowledgements

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“I’d like to thank the Academy...”

Acknowledgements

This research would not have been possible without these phenomenal people.

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- Mount Holyoke College Summer REU Program

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