Selfatopes and their Inner Normal Fans

An Honors Thesis

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Submitted

December 2005

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Introduction

Toric varieties arise in the study of computational algebraic geometry and provide good examples of algebraic manifolds. Many properties of toric varieties can be determined from a study of their cell structure. While this structure can be very difficult to compute for a general algebraic variety, the cell structure of toric varieties is related to certain types of polyhedra from convex geometry. This is useful because the properties of polyhedra are intuitive and relatively easy to compute with mundane linear algebra.

In this thesis we prove a uniqueness condition for the class of polyhedra which give rise to smooth projective toric varieties.

Main Theorem. *Up to translation, a polytopal fan can be the inner normal fan of at most one smooth lattice polytope whose edges contain no interior lattice points.*

The first section of this paper develops the ideas of convex geometry necessary for a basic discussion of polytopes and cones, while the second section is concerned with the combinatorial structure of these objects. The third section introduces the concept of duality and presents several results which allow us to compute generators for dual cones. In the section about fans, the ideas from the previous sections are brought together to define the inner normal fan of a polytope. The concept of smoothness is presented and developed in the next section and it is here that selfatopes are defined. The main theorem appears at the end of the last section.

Convex Polyhedra

The purpose of this section is to familiarize the reader with the core concepts and basic objects of convex geometry. We define polytopes and polyhedra leading up to the statement of the fundamental theorem of polytopes. The section concludes with a lemma concerning the intersection of affine hyperplanes, which will be of importance in all of the following sections.

Definition 1.1. A set A is *convex* if for every pair of elements p, $q \in A$ we have $tp + (1-t)q \in A$ for every $t \in [0, 1]$. Interpreted geometrically, we say that A is convex if it contains the line segment $\overline{pq}$ for every pair of elements of p, $q \in A$.

Example 1.2. Figure 1 shows a picture of three subsets of the plane. The sets A and B are examples of convex sets. However, the set on the right is not convex because it does not contain the line segment between its two upper vertices.

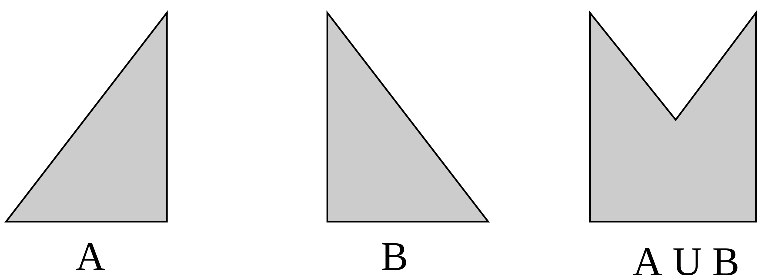

Figure 1: Two convex sets and one non-convex set.

Definition 1.3. Let $\{a_1, \ldots, a_k\} \subseteq \mathbb{R}^n$ be any finite collection of points. A linear combination such as

$$x = r_1 a_1 + \ldots + r_k a_k, \qquad r_i \in \mathbb{R}$$

is called a *convex combination* if the coefficients satisfy

$$r_1 + \ldots + r_k = 1 \qquad \text{and} \qquad r_i \geq 0 \ , \quad \forall\, i.$$

If $A \subseteq \mathbb{R}^n$, then the *convex hull of* A is the collection of all convex combinations of elements of A. We write

$$\mathrm{conv}(A) = \{\sum_{i=1}^{k} r_i a_i \mid a_i \in A,\ r_i \in \mathbb{R}_{\geq 0},\ \sum_{i=1}^{k} r_i = 1\ \}.$$

It is not difficult to see that the convex hull of A is the smallest convex set which contains A [5][pages 3-4].

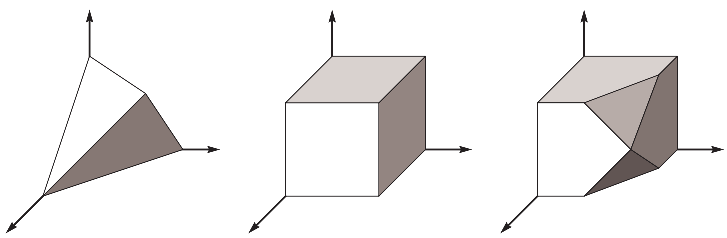

Figure 2: Three 3-dimensional polytopes.

Definition 1.4. A *polytope* is the convex hull of a finite set of points.

Example 1.5. Figure 2 shows three simple examples of 3-dimensional polytopes. The coordinate axes are shown for reference. Figure 3 shows an example of a convex hull. The set on the right is the convex hull of the set A on the left.

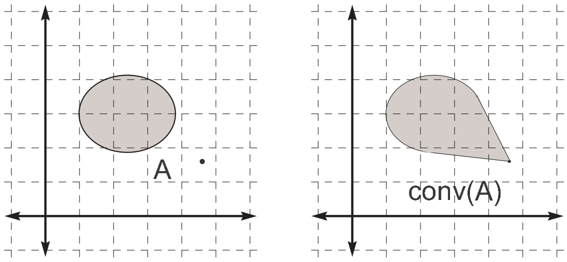

Figure 3: A set and its convex hull

Definition 1.6. Let $a \in \mathbb{R}^n$ be a non-zero vector and let $\alpha \in \mathbb{R}^n$ be any real number. An *affine hyperplane* is a set of the form

$$H = \{\, x \in \mathbb{R}^n \mid \langle a,\, x\rangle = \alpha \,\}.$$

In this form, the defining vector a is called the *normal vector* of the hyperplane H. Every affine hyperplane defines two *half-spaces* of the form

$$H^+ = \{\, x \in \mathbb{R}^n \mid \langle a,\, x\rangle \geq \alpha \,\}$$
$$H^- = \{\, x \in \mathbb{R}^n \mid \langle a,\, x\rangle \leq \alpha \,\}$$

There are several things to note. First, if $\alpha = 0$, then $H = a^\perp$ is the orthogonal complement of the linear space spanned by the vector a

and H is an $n-1$ dimensional linear subspace of $\mathbb{R}^n$. In general, every affine hyperplane is a translate of some such subspace.

Second, the normal vector a points into the positive half-space but points out of the negative half-space (See Figure 4 below). For this reason, the normal vector a of H is called the *inner normal vector* to the half-space H^+, but the *outer normal vector* to the half-space H^-.

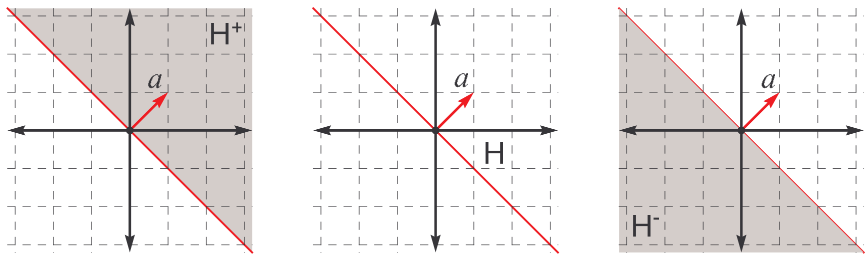

Figure 4: An affine hyperplane and two half-spaces.

Definition 1.7. A *polyhedral set* or *polyhedron* is the intersection of a finite number of half-spaces. Moreover, since each half-space is the solution set of a linear inequality, a polyhedron is the solution set of a linear system of inequalities.

$$P = \bigcap_{j=1}^{k} H_j^+ \; = \{ \, x \in \mathbb{R}^n \mid Ax \geq \alpha \, \}$$

In the expression above, A is the $k \times n$ real matrix whose rows are the inner normal vectors to these half-spaces.

$$A = \begin{pmatrix} - & a_1 & - \\ & \vdots & \\ - & a_k & - \end{pmatrix}$$

Later on we will give a definition of the dimension of a polyhedron. But intuitively, we say that P is k-dimensional and write $\dim(P) = k$ if P can be expressed as subset of $\mathbb{R}^k$, but not as a subset of $\mathbb{R}^{k-1}$.

Lemma 1.8. *Polyhedra are closed sets.*

Proof. All half-spaces are closed and the intersection of a finite collection of closed sets is itself a closed set. Therefore, every polyhedron is closed. □

In this paper, we will be interested in two types of polyhedra. The first type is the polytope, which we have already discussed. The second type is a polyhedral cone.

Definition 1.9. A non-empty subset $\sigma \subseteq \mathbb{R}^n$ is called a *cone* if whenever $x \in \sigma$ and $r \in \mathbb{R}_{\geq 0}$ we have $rx \in \sigma$.

For the remainder of the thesis, we will concern ourselves with *convex polyhedral cones.* These are cones which are also convex polyhedra. Convex polyhedral cones can be expressed as the *positive hull* of a finite set of vectors. For example,

$$\sigma = \text{pos}\{a_1, \ldots, a_k\} = \{\sum_{i=1}^{k} r_i a_i \mid r_i \in \mathbb{R}_{\geq 0}\}.$$

The vectors $a_1, \ldots, a_k$ are called *generators* of the cone σ and the set $A = \{a_1, \ldots, a_k\}$ is called a *generating set.* If σ cannot be generated by a proper subset of A, then we say that A is a *minimal* generating set.

Several characteristics of cones will be important for later topics. For instance, if $\sigma = \operatorname{pos} A$ for some set $A \subseteq \mathbb{R}^n$ of generators, then we define the *dimension* of σ to be the vector space dimension of $\operatorname{span} A$. A cone is called *pointed* if it does not contain a subspace of dimension greater than zero.

We will distinguish cones by the algebraic properties of their generators. The subset $\mathbb{Z}^n \subseteq \mathbb{R}^n$ is called the *lattice* and any point in $\mathbb{Z}^n$ is referred to as a *lattice point*. Let $v = (v_1, \ldots, v_n) \in \mathbb{Z}^n$ be a lattice point. If $\gcd(v_1, \ldots, v_n) = 1$, then we say that v is *primitive*.

A cone that can be expressed as the positive hull of a finite set of lattice points is called a *rational* cone.

Example 1.10. Consider the two planar cones σ and τ below.

$$\sigma = \operatorname{pos}\{\, \left(\begin{smallmatrix}3\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}2\\3\end{smallmatrix}\right) \} \qquad \tau = \operatorname{pos}\{\, \left(\begin{smallmatrix}\pi\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}2\\e\end{smallmatrix}\right) \}$$

Clearly, σ is rational. The cone τ is not rational because it cannot be expressed as the positive hull of a finite set of lattice points.

Observe that both σ and τ are pointed cones. For an example of a cone that is not pointed, consider the half-space on the right hand side

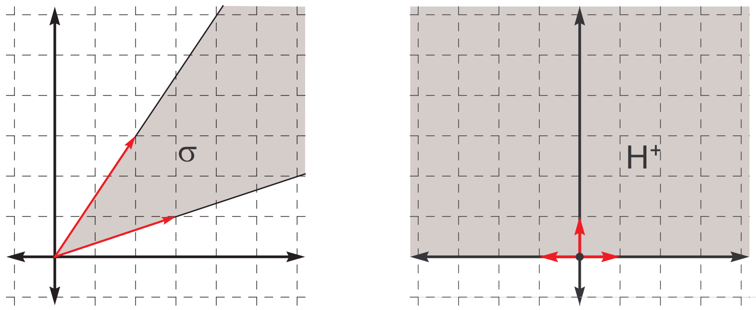

Figure 5: Pointed and non-pointed cones.

of Figure 5. This half-space is also rational since

$$H^{+} = \{\, x \in \mathbb{R}^2 \mid \langle \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right), x \rangle \geq 0 \,\}$$
$$= \operatorname{pos}\{ \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right) \}.$$

Lemma 1.11. *If σ is a rational convex polyhedral cone, then it has a* primitive generating set *or a generating set whose elements are primitive.*

Proof. Suppose that $A = \{a_1, \ldots, a_k\} \subseteq \mathbb{Z}^n \setminus \{0\}$ so that $\sigma = \operatorname{pos} A$ is a rational convex polyhedral cone. Write each of the generators in the expanded form

$$a_j = (a_{j1}, \ldots, a_{jn}) \in \mathbb{Z}^n$$

and define $d_j = \gcd(a_{j1}, \ldots, a_{jn})$. Note that $d_j \neq 0$ since A does not contain the origin. For each $a_j \in A$ define the vector $b_j = (\frac{1}{d_j})a_j$. Clearly each of these vectors are primitive.

We have $a_j = d_j b_j$ and $b_j = (\frac{1}{d_j})a_j$. Therefore, $A \subseteq \operatorname{pos} B$ and

$B \subseteq \operatorname{pos} A$. This obviously implies that

$$\sigma = \operatorname{pos} A = \operatorname{pos} B$$

and hence the set $B = \{b_1, \ldots, b_k\}$ is a primitive generating set for σ. □

It is natural to ask when a polytope can be expressed in polyhedral form and vice-versa. The Fundamental Theorem of Polyhedra (stated below) answers this question. We omit the proof of the fundamental theorem because a discussion of the methods involved would lead us too far astray. For a detailed proof, the interested reader is referred to Ziegler's book [5][Chapter 1]. For our purposes, an example will suffice.

Theorem 1.12 (Fundamental Theorem of Polyhedra)**.** *[5] Every bounded polyhedron is a polytope and every polytope is a bounded polyhedron. Also, every convex polyhedral cone has a finite generating set.*

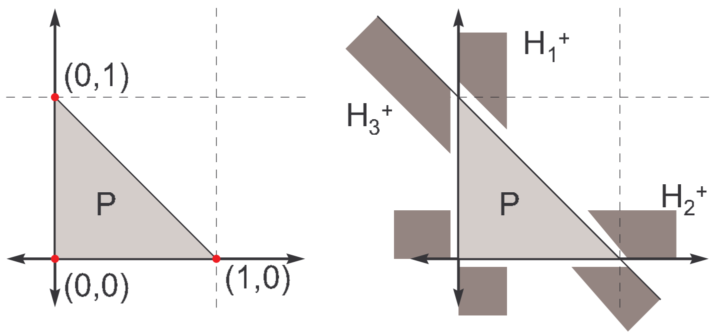

Figure 6: The 2-simplex as a polytope and a polyhedron.

Example 1.13. Figure 6 shows the standard 2-dimensional simplex

$$P = \text{conv}\{\,(0,0),\ (1,0),\ (0,1)\,\} \subseteq \mathbb{R}^2.$$

The 2-simplex P is a polytope since it is clearly the convex hull of a finite point-set. On the other hand, P is also a polyhedron since we can express it as the intersection of three half-spaces. In particular, if we define the half-spaces

$$\begin{aligned} H_1^+ &= \{\,(x,y) \in \mathbb{R}^2 \mid x \geq 0\,\} \\ H_2^+ &= \{\,(x,y) \in \mathbb{R}^2 \mid y \geq 0\,\} \\ H_3^+ &= \{\,(x,y) \in \mathbb{R}^2 \mid -x-y \geq -1\,\}, \end{aligned}$$

then we can express P in the form

$$\begin{aligned} P &= H_1^+ \cap H_2^+ \cap H_3^+ \\ &= \{\,\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in \mathbb{R}^2 \mid \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{smallmatrix}\right) \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \geq (0,0,-1)\,\} \end{aligned}$$

For several results in the following sections, we will be concerned about sets formed by the intersection of several hyperplanes. The following Lemma tells us when the intersection of several hyperplanes is non-empty and what the dimension of that intersection is.

Let $\{H_1, \dots H_k\}$ be a collection of hyperplanes of the form

$$H_j = \{ \, x \in \mathbb{R}^n \mid \langle a_j, x\rangle = \alpha_j \, \}.$$

Define A to be the $k \times n$ real matrix whose rows are the normal vectors to these hyperplanes.

Lemma 1.14. *If the set of vectors $\{a_1, \dots, a_k\}$ is linearly independent, then the intersection*

$$I = \bigcap_{j=1}^{k} H_j$$

is non-empty and $I = h + \ker A$ for some $h \in I$.

Proof. First compute

$$I = \, \{ \, x \in \mathbb{R}^n \mid Ax = \alpha \, \}.$$

We can think of the matrix A as the matrix of a linear transformation $A : \mathbb{R}^n \to \mathbb{R}^k$. Since the rows of A are independent, the rank of A is k and consequently the map is surjective. This implies that the intersection is non-empty since the defining equation $Ax = \alpha$ will always have a solution.

Let $h \in I$ and compute

$$
\begin{aligned}
(-h) + I &= \{\ (x - h) \in \mathbb{R}^n \mid Ax = \alpha\ \} \\
&= \{\ x \in \mathbb{R}^n \mid A(x + h) = \alpha\ \} \\
&= \{\ x \in \mathbb{R}^n \mid Ax + Ah = \alpha\ \} \\
&= \{\ x \in \mathbb{R}^n \mid Ax + \alpha = \alpha\ \} \\
&= \{\ x \in \mathbb{R}^n \mid Ax = 0\ \} \\
&= \ker\ A
\end{aligned}
$$

Thus $I = h + \ker\ A$. □

Lemma 1.15. *If the intersection $I = \bigcap_{j=1}^{k} H_j$ is non-empty, then dim(I) = rank(A) = $n - k$.*

Proof. Let $h \in I \neq \emptyset$. By Lemma 1.14 above, it is clear that $I = h + \ker A$. By applying the Rank-Nullity Theorem[2][page 45], it is clear that

$$
\begin{aligned}
\dim(\mathbb{R}^n) &= \dim(\mathrm{Im}\ A) + \dim(\ker\ A) \\
n &= k\ +\ \dim(\ker\ A) \\
n - k &= \dim(\ker\ A).
\end{aligned}
$$

Therefore, $\dim(I) = \dim(h + \ker\ A) = \dim(\ker\ A) = n - k$. □

Combinatorics

The combinatorial structure of a polyhedron is, roughly, how the polyhedron can be put together from smaller polyhedra. The goal of this section is to introduce and explain the combinatorial structure of polytopes and cones and to provide a few basic results.

Definition 2.1. Let $P \subseteq \mathbb{R}^n$ be a polyhedron and suppose that $H \subseteq \mathbb{R}^n$ is a hyperplane. If $H \cap P \neq \emptyset$ and $P \subseteq H^+$, then $F = H \cap P$ is a *face* of the polyhedron P. We will say that H is a *supporting hyperplane for* the face F or H is a *supporting hyperplane of* the polyhedron P. Which description we use will depend on the specific emphasis needed.

We always consider P a face of itself, since we could embed $\mathbb{R}^n$ as a hyperplane $H \subseteq \mathbb{R}^{n+1}$ which would give us $P = H \cap P$. For technical reasons, we will consider the empty set to be a face of a polytope. Faces which are neither the entire polyhedron, nor the empty set are referred to as *proper faces.*

Example 2.2. Figure 7 shows a planar polytope with two supporting hyperplanes. The positive half-spaces H_1^+ and H_2^+ are shown shaded. The hyperplane H_1 intersects P in a face F of dimension 1, while H_2 intersects P in a point p. Therefore, both F and p are proper faces of

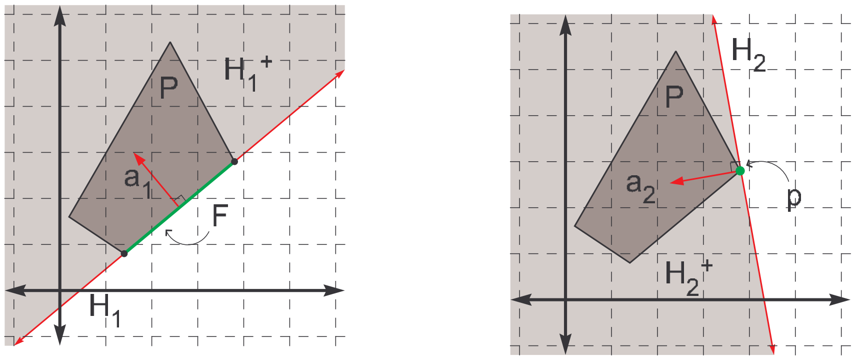

Figure 7: Proper faces of a polytope.

the polytope P.

$$H_1 = \{\, x \in \mathbb{R}^2 \,|\, \langle a_1, x\rangle = k_1 \,\}$$

$$\text{and} \qquad H_2 = \{\, x \in \mathbb{R}^2 \,|\, \langle a_2, x\rangle = k_2 \,\}$$

Lemma 2.3. *A face of a polyhedron is a polyhedron.*

Proof. Let $P \subseteq \mathbb{R}^n$ be a polyhedral set and suppose that H is a supporting hyperplane of the face F. By definition, the polyhedron P is the intersection of finitely many half-spaces. Let $H_1^+, \dots, H_m^+$ be half-spaces such that

$$P = \bigcap_{j=1}^{m} H_j^+ \; .$$

Then we can compute

$$\begin{aligned} F &= P \cap H \\ &= (\bigcap_{j=1}^{m} H_j^+) \cap (H^+ \cap H^-) \end{aligned}$$

and see that F is the intersection of a finite collection of half-spaces. Therefore, F is a polyhedron. $\square$

Lemma 2.4. *Let $P \subseteq \mathbb{R}^n$ be an n-dimensional polyhedron. If F is a face of P and G is a face of F, then G is a face of P.*

Proof. Suppose that G is a face of F and F is a face of P. Without loss of generality, assume that $0 \in G$. Then there exist supporting hyperplanes

$$\begin{aligned} & H_G = \{\, x \in \mathbb{R}^n \mid \langle a_G, x \rangle = 0 \,\} \\ \text{and} \qquad & H_F = \{\, x \in \mathbb{R}^n \mid \langle a_F, x \rangle = 0 \,\}, \end{aligned}$$

such that $G = F \cap H_G$ and $F = P \cap H_F$. Furthermore, every supporting hyperplane of P for F is also a supporting hyperplane of P for G because $G \subseteq F = H_F \cap P$.

Choose a set of generators $\{v_1, \ldots, v_k\}$ for $\operatorname{pos} P$. If necessary, we can re-index the generators and assume that the first j generators are the ones that lie in F.

Since H_F is a supporting hyperplane, we have $P \subseteq H_F^+$ and $F \subseteq H$, hence $\langle p, a_F \rangle > 0$ for all $p \in P \setminus F$. In particular, $\langle v_i, a_F \rangle > 0$ whenever $j < i \leq k$. Therefore, there must exist a positive real number $\epsilon > 0$ such that $\langle v_i, a_G + \epsilon a_F \rangle > 0$ for all $j < i \leq k$.

We claim that

$$H = \{ x \in \mathbb{R}^n \mid \langle x, a_G + \epsilon a_F \rangle = 0 \}$$

is a supporting hyperplane for $G \subseteq P$. Clearly, $P \subseteq H^+$ since for every generator v_i of P we have $\langle v_i, a_G + \epsilon a_F \rangle \geq 0$. Furthermore, if $g \in G \subseteq F$, then

$$\langle g, a_G + \epsilon a_F \rangle = \langle g, a_G \rangle + \epsilon \langle g, a_F \rangle = 0$$

which implies $G \subseteq P \cap H$. □

As we see in Definition 2.5 below, some faces have special names.

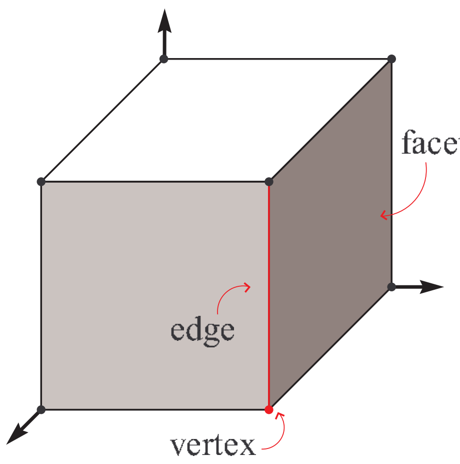

Figure 8: A vertex, an edge and a facet.

Definition 2.5. Let P be an n-dimensional polyhedron. A *facet* of P is an $n-1$ dimensional face. A *vertex* is a 0-dimensional face and an *edge* is a 1-dimensional face. We say that two vertices p and q are *adjacent* if they are contained in a common edge E of P. In that case, we have $E = \operatorname{conv}\{\,p,\, q\,\}$. Occasionally, we will use $\operatorname{vert}(P)$ to denote the collection of all vertices of a polyhedron.

Now that we have Definition 2.5, we can begin to describe how the vertices and edges of polytopes fit together.

Lemma 2.6. *If P is an n-dimensional polytope, then every vertex of P is adjacent to at least n vertices of P.*

Proof. Suppose P is an n-dimensional polytope and let $p \in P$ be a vertex. Denote the vertices adjacent to p by $w_1, \ldots, w_k$ so that

$$W = \{(w_1 - p), \ldots, (w_k - p)\}$$

is the set of all edge vectors at the vertex $p \in P$. Next, define the cone

$$\sigma(p) = \operatorname{pos}\{(w_1 - p), \ldots, (w_k - p)\}$$

Observe that $P \subseteq p + \sigma(p)$. Since P is n-dimensional, $\sigma(p)$ must be n-dimensional too. This implies that $\operatorname{span} W = \mathbb{R}^n$ and hence W must contain a (vector space) basis for $\mathbb{R}^n$. Thus $k \geq n$ which means that p is contained by at least n edges. Therefore, each vertex of P is adjacent to at least n vertices. □

Theorem 2.7. *[1][Theorem 1.11, page 33] If P be a polyhedral set,*

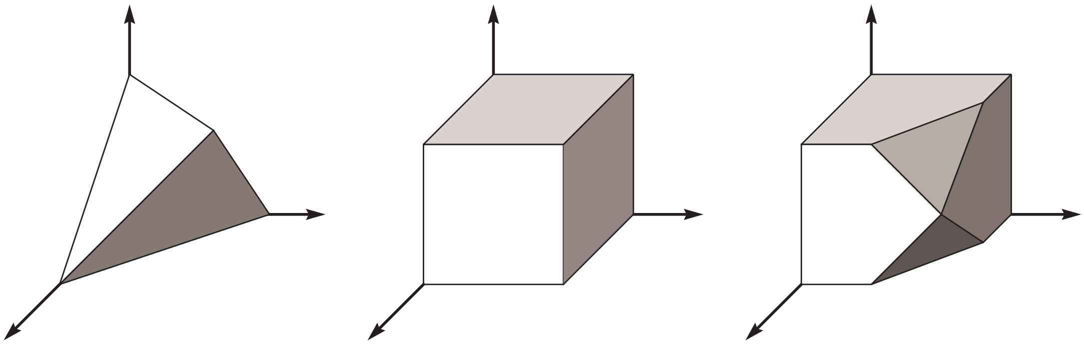

Figure 9: Vertices of 3-dimensional polytopes are contained in at least three edges.

then every proper face $\emptyset \neq E \subsetneq P$ *is the intersection of the facets which contain* E.

Although we omit the proof, the Theorem 2.7 will be important in the last section.

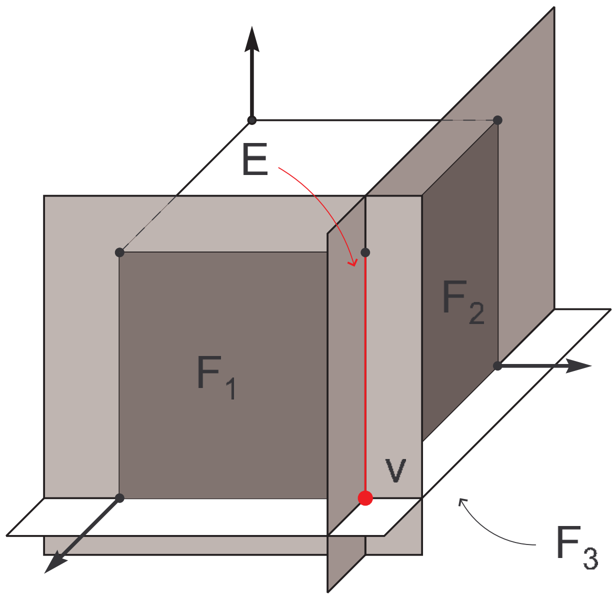

Figure 10: An edge and a vertex as the intersection of facets.

Example 2.8. As an illustration of Theorem 2.7, consider the unit 3-

cube in Figure 10. The picture shows three facets embedded in their respective supporting hyperplanes. The facet F_3 is obscured. From this picture, it is obvious that the vertex $(1, 1, 0)$ is the intersection of all three facets F_1, F_2 and F_3, while the edge E is the locus of points common to F_1 and F_2.

Definition 2.9. A *partially ordered set* or *poset* is a set X together with a relation " $\prec$ ", called a *partial ordering*, such that for all a, b, $c \in X$ we have

- Reflexivity: $a \prec a$
- Anti-symmetry: $a \prec b$ and $b \prec a$ implies $a = b$
- Transitivity: $a \prec b$ and $b \prec c$ implies $a \prec c$

Now we want to construct a poset which is important to the current discussion. If P is a polyhedron, we denote the collection of all of its faces by $\mathcal{F}(P)$. For any G, $F \in \mathcal{F}(P)$ we write $G \prec F$ if G is a face of F. We call $\mathcal{F}(P)$ together with the ordering $\prec$ the *face poset of P*.

Proposition 2.10. *The face poset of a polyhedron P is indeed a poset.*

Proof. It is sufficient to verify that the reflexive, anti-symmetric and transitive properties of the $\prec$ relation are satisfied on the set $\mathcal{F}(P)$.

The reflexive property clearly holds because every polyhedron is a face of itself. Therefore, for each face $E \in \mathcal{F}(P)$, we have $E \prec E$.

To verify anti-symmetry, observe that $E \prec F$ implies $E \subseteq F$ for every pair of faces E, $F \in \mathcal{F}(P)$. Thus if $E \prec F$ and $F \prec E$, we certainly have $E \subseteq F$ and $F \subseteq E$. Therefore $E = F$.

Last of all, suppose that E, F, $G \in \mathcal{F}(P)$ are faces such that $E \prec F$ and $F \prec G$. By Lemma 2.4, we have $E \prec G$. □

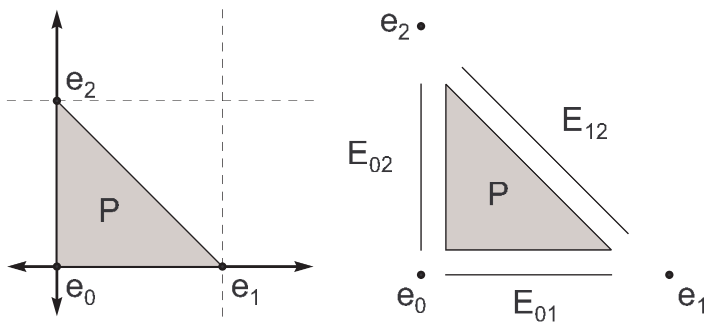

Figure 11: The faces of a 2-simplex.

Example 2.11. Consider the 2-simplex in Figure 11 below.

$$P = \text{conv}\{(0,0),\ (1,0),\ (0,1)\}$$

The face poset $\mathcal{F}(P)$ has a total of eight elements.

$$\mathcal{F}(P) = \{\,\emptyset,\ e_0,\ e_1,\ e_2, E_{01},\ E_{02},\ E_{12},\ P\}$$

The proper faces of P are defined as follows.

$$
\begin{array}{ll}
e_0 = \{(0,0)\} & E_{01} = \text{conv}\{\,(0,0),\ (1,0)\,\} \\
e_1 = \{(1,0)\} & E_{02} = \text{conv}\{\,(0,0),\ (0,1)\,\} \\
e_2 = \{(0,1)\} & E_{12} = \text{conv}\{\,(1,0),\ (0,1)\,\}
\end{array}
$$

Definition 2.12. Let $(P, \prec)$ and $(Q, \lhd)$ be posets. A *combinatorial morphism* is a function

$$\varphi : P \longrightarrow Q$$

which preserves the partial ordering. That is,

$$p_1 \prec p_2 \quad \Longrightarrow \quad \varphi(p_1) \lhd \varphi(p_2)$$

If, in addition, φ is bijective, then we say that it is a *combinatorial isomorphism*.

Example 2.13. Let P be the 2-simplex in Figure 11 above. Observe that the power set $\mathcal{P}(\{0,\ 1,\ 2\,\}$ together with the inclusion relation $\subseteq$ is a poset. Let

$$\varphi : \mathcal{F}(P) \longrightarrow \mathcal{P}(\{0,\ 1,\ 2\,\}$$

be the bijective function diagramed below.

$$
\begin{array}{lcl}
\emptyset & \longrightarrow & \emptyset \\
\{(0,0)\} & \longrightarrow & \{0\} \\
\{(1,0)\} & \longrightarrow & \{1\} \\
\{(0,1)\} & \longrightarrow & \{2\} \\
\text{conv}\{\,(0,0),\ (1,0)\,\} & \longrightarrow & \{0,\ 1\} \\
\text{conv}\{\,(0,0),\ (0,1)\,\} & \longrightarrow & \{0,\ 2\} \\
\text{conv}\{\,(0,0),\ (1,0)\,\} & \longrightarrow & \{1,\ 2\} \\
\text{conv}\{\,(0,0),\ (1,0)\,(0,1)\,\} & \longrightarrow & \{0,\ 1,\ 2\}
\end{array}
$$

By inspection, it is clear that φ is a combinatorial isomorphism since $E \prec F$ implies $\varphi(E) \subseteq \varphi(F)$ for every pair of faces $E,\ F \in \mathcal{F}(P)$.

Duality

The concept of duality is essential to the study of convex geometry and is an important part of the main topic of this thesis. In this section we define the dual cones and present some important facts about them.

Definition 3.1. Let $\sigma \subseteq \mathbb{R}^n$ be a convex cone. We define the *dual cone* of σ to be the cone

$$\sigma^\vee = \{\, x \in \mathbb{R}^n \mid \langle u, x \rangle \geq 0,\ \forall\, u \in \sigma\}$$

Lemma 3.2. *Let $\sigma \subseteq \mathbb{R}^n$ be a convex cone. Then*

$$(\sigma^\vee)^\vee = \sigma$$

Proof. First, suppose that $x \in \sigma$. Then for every $y \in \sigma^\vee$ we have

$$\langle x, y \rangle = \langle y, x \rangle \geq 0.$$

Therefore, $\sigma \subseteq (\sigma^\vee)^\vee$.

On the other hand, if $x \notin \sigma$, then there must exist a $y \in \sigma^\vee$ such that $\langle y, x \rangle < 0$. See [3][page 9] or [4][page 11]. Therefore, $x \notin \sigma$ implies $x \notin (\sigma^\vee)^\vee$ and hence $(\sigma^\vee)^\vee \subseteq \sigma$. □

Lemma 3.3. *Let* $\sigma = \{\, x \in \mathbb{R}^n \mid Ax \geq 0 \,\}$ *be a pointed* n*-dimensional convex polyhedral cone, where the rows of the matrix*

$$A = \begin{pmatrix} - & a_1 & - \\ & \vdots & \\ - & a_k & - \end{pmatrix}$$

are the inner normal vectors to the facets of σ*. Then* $\sigma^\vee = \operatorname{pos}\{\, a_1, \ldots, a_k \,\}$*.*

Proof. For brevity, denote $\mathcal{A} = \{\, a_1, \ldots, a_k \,\}$. Observe that for each $a \in \mathcal{A}$ the hyperplane

$$H = \{\, x \in \mathbb{R}^n \mid \langle a, x \rangle = 0\}$$

is a supporting hyperplane of σ. This implies that for each vector $a \in \mathcal{A}$, we have $\langle u, a \rangle \geq 0$ for all $u \in \sigma$. Therefore,

$$\mathcal{A} \subseteq \{\, x \in \mathbb{R}^n \mid \langle u, x \rangle \geq 0, \ \forall\, u \in \sigma\} = \sigma^\vee.$$

Hence, $\sigma^\vee \supseteq \operatorname{pos} \mathcal{A}$.

On the other hand, suppose that $x \notin \operatorname{pos} \mathcal{A}$. Since σ is pointed, it is clear that

$$\ker \begin{pmatrix} - & a_1 & - \\ & \vdots & \\ - & a_k & - \end{pmatrix} = \{0\}.$$

Thus, $\operatorname{span} \mathcal{A} = \mathbb{R}^n$ and hence $\mathcal{A}$ contains a basis for $\mathbb{R}^n$. Without loss of generality, we can assume that the first n elements of $\mathcal{A}$ form this basis. Then there are real numbers $\gamma_1, \ldots, \gamma_n$ such that

$$x = \gamma_1 a_1 + \cdots + \gamma_n a_n.$$

Clearly, $x \neq 0$ since $0 \in \operatorname{pos} \mathcal{A}$, so not all of the coefficients are zero. However, there must exist at least one negative coefficient because we have assumed

$$x \notin \operatorname{pos}\{ a_1, \ldots, a_n \} \subseteq \operatorname{pos} \mathcal{A}.$$

Let us partition the set of coefficients $\{\gamma_1, \ldots, \gamma_n\}$ into two subsets. Define

$$\{-\beta_1, \ldots, -\beta_j\} = \{ \gamma_i \mid \gamma_i < 0 \}$$
$$\text{and} \quad \{\beta_{j+1}, \ldots, \beta_n\} = \{ \gamma_i \mid \gamma_i \geq 0 \}.$$

Up to a re-labeling of the vectors, we can write

$$x = -(\beta_1 a_1 + \cdots + \beta_j a_j) + (\beta_{j+1} a_{j+1} + \cdots + \beta_n a_n)$$

where $\beta_i \in \mathbb{R}_{\geq 0}$ for all $1 \leq i \leq r$.

We claim that the vectors $\{ a_{j+1}, \ldots, a_n \}$ must generate a face of $\operatorname{pos} \mathcal{A}$. To see this, let $\epsilon \geq 0$ be any non-negative real number and define

$$x_\epsilon := -\epsilon(\beta_1 a_1 + \cdots + \beta_j a_j) + (\beta_{j+1} a_{j+1} + \cdots + \beta_n a_n).$$

Clearly, $x_\epsilon \notin \operatorname{pos}\mathcal{A}$ whenever $\epsilon > 0$. On the other hand,

$$x_0 = (\beta_{j+1}a_{j+1} + \cdots + \beta_n a_n) \in \operatorname{pos}\mathcal{A}$$

Therefore, $\operatorname{span}\{\, a_{j+1},\, \ldots,\, a_n \,\}$ lies entirely on the boundary of $\operatorname{pos}\mathcal{A}$. However, the boundary of the $\operatorname{pos}\mathcal{A}$ is the union of all its proper faces [1][pages 33-34]. Hence $\operatorname{span}\{\, a_{j+1},\, \ldots,\, a_n \,\}$ is a face of $\operatorname{pos}\mathcal{A}$.

Therefore, there exists a supporting hyperplane for this face and in particular, there is a vector $v \in \sigma$ such that

$$\langle v, a_{j+1}\rangle = \cdots = \langle v, a_n\rangle = 0\,,$$

but $\langle v, a_i\rangle > 0$ for all $1 \le i \le j$. This implies that

$$\begin{aligned}\langle v, x\rangle &= -\langle v, (\beta_1 a_1 + \cdots + \beta_j a_j)\rangle + \langle v, (\beta_{j+1}a_{j+1} + \cdots + \beta_n a_n)\rangle \\ &= -\langle v, (\beta_1 a_1 + \cdots + \beta_j a_j)\rangle \\ &< 0\,.\end{aligned}$$

Therefore,

$$x \notin \sigma^\vee = \{\, x \in \mathbb{R}^n \,|\, \langle v, x\rangle \ge 0,\ \forall\, v \in \sigma\}.$$

This shows that $x \notin \operatorname{pos}\{\, a_1,\, \ldots,\, a_k \,\}$ implies $x \notin \sigma^\vee$, and this is equivalent to $\sigma^\vee \subseteq \operatorname{pos}\{\, a_1,\, \ldots,\, a_k \,\}$. $\square$

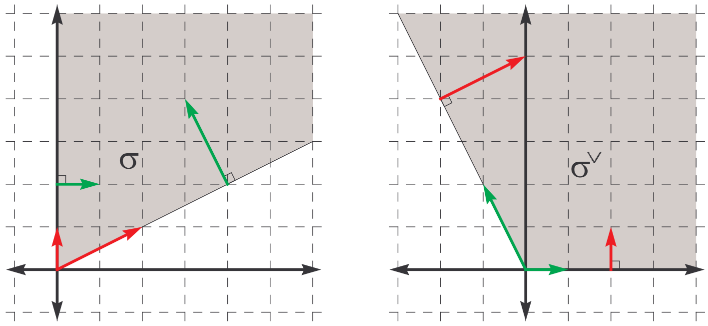

Figure 12: A cone and its dual.

Example 3.4. Consider the cone $\sigma \subseteq \mathbb{R}^2$. See Figure 12.

$$\sigma = \operatorname{pos}\{ \left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \}$$

The facets of σ are the two edges of σ. Since the edges are each generated by a single vector, the inner normal to a given facet is the vector normal to its generator which points towards the interior of σ. Therefore,

$$\sigma^\vee = \operatorname{pos}\{ \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 \\ 2 \end{smallmatrix}\right) \}.$$

Observe also that for the cones in this example, we have the equality $(\sigma^\vee)^\vee = \sigma$, as in Lemma 3.2.

Corollary 3.5. *If $a_1, \ldots, a_k$ are the inner normal vectors to the facets of a n-dimensional convex polyhedral cone $\sigma \subseteq \mathbb{R}^n$, then $\sigma^\vee = \operatorname{pos}\{ a_1, \ldots, a_k \}$.*

Proof. If $a_1, \ldots, a_k$ are the inner normal vectors to the facets of an n-dimensional cone $\sigma \subseteq \mathbb{R}^n$, then σ can be expressed in polyhedral form

$$\sigma = \{\, x \in \mathbb{R}^n \mid \begin{pmatrix} - & a_1 & - \\ & \vdots & \\ - & a_k & - \end{pmatrix} x \geq 0 \,\}.$$

From here, the result follows from Lemma 3.3. $\square$

Fans

In this section, we describe how to construct polyhedral cones from the faces of a polytope in a way that encodes the structure of the polytope. The collection of cones we get from this process is called a fan and plays a key role in the remaining sections.

Definition 4.1. Let $P \subseteq \mathbb{R}^n$ be an n-dimensional polytope and suppose that $F \subseteq P$ is a face. We define the *inner normal cone* of the face F to be the set

$$\sigma_F = \{\, x \in \mathbb{R}^n \mid \langle u - v,\, x\rangle \geq 0\,, \forall u \in P,\, \forall v \in F\,\}.$$

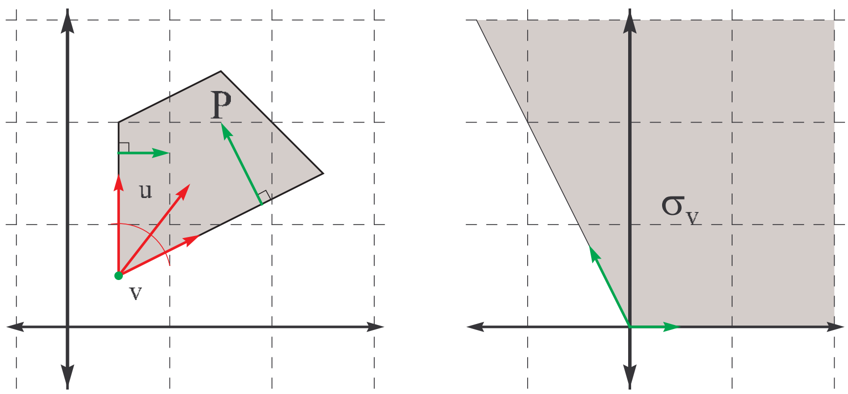

Figure 13: The inner normal cone of a vertex.

Example 4.2. Figure 13 shows a polytope P with a vertex $v \in P$. Illustrated in red are some vectors of the form $u - v$ where u is some point in P. From the definition of the inner normal cone, σ_v is the intersection of all positive half-spaces through the origin with inner normal vector of the form $u - v$.

$$\sigma_v = \bigcap_{u \in P} \{\, x \in \mathbb{R}^2 \mid \langle u - v, x \rangle \geq 0 \,\}$$

The cone σ_v is shown to the right. Observe that in this particular case, σ_v is generated by the inner normal vectors of the facets which contain v.

Lemma 4.3. *Let $P \subseteq \mathbb{R}^n$ be a full dimensional polytope and suppose $E \subseteq P$ is a proper face. If $\{F_1, \ldots, F_k\}$ denotes the set of all facets of P which contain E, then*

$$\sigma_E = \mathrm{pos}\{a_1, \ldots, a_k\}$$

where a_j denotes the inner normal vector to the facet F_j.

In the next section, we will require the following result about the inner normal cones of vertices.

Lemma 4.4. *Let P be a polytope with $p \in P$ a vertex. For each edge $E_i \subseteq P$ which contains p, define w_i to be the closest lattice point to p in the edge E_i. Then*

$$\sigma_p^{\vee} = \mathrm{pos}\{\, (w_1 - p), \ldots, (w_n - p) \,\}$$

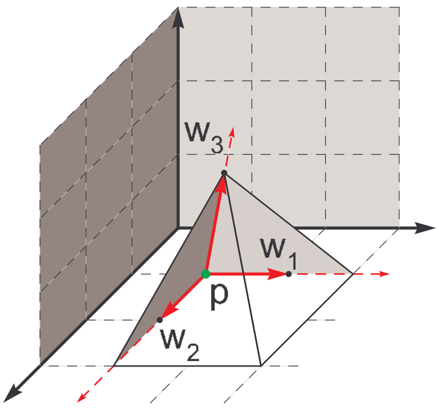

Figure 14: Edge vectors emanating from a vertex.

Proof. Let p be a vertex of the polytope P and define

$$\tau = \text{pos}\{\, (w_1 - p),\, \ldots,\, (w_n - p)\, \}.$$

Since $\text{conv}(S) \subseteq \text{pos}(S)$, both τ and $\text{pos}(-p + P)$ are generated by the edge vectors $\{\, (w_1 - p),\, \ldots,\, (w_n - p)\, \}$. See Figure 14 above. Therefore, $\tau = \text{pos}(-p + P)$ and hence the facets of τ have the same inner normal vectors as the facets of P which contain p. Suppose that $a_1,\, \ldots,\, a_k$ are these inner normal vectors. By applying Lemma 4.3 and Corollary 3.5 we see that

$$\sigma_p = \text{pos}\{a_1,\, \ldots,\, a_k\,\} = \tau^\vee.$$

Finally, Lemma 3.2 implies

$$\sigma_p^\vee = \tau = \text{pos}\{\, (w_1 - p),\, \ldots,\, (w_n - p)\, \}.$$

□

Corollary 4.5. *Let E and F be faces of a n-dimensional polytope $P \subseteq \mathbb{R}^n$. If $E \subseteq F$, then $\sigma_F \subseteq \sigma_E$.*

Proof. If $E \subseteq F$, then every facet which contains F must also contain E. Therefore, every generator of σ_F is contained in σ_E. Hence $\sigma_F \subseteq \sigma_E$. □

Definition 4.6. A *fan* $\triangle$ is a collection of cones in $\mathbb{R}^n$ which satisfy

- If $\tau \prec \sigma$ and $\sigma \in \triangle$, then $\tau \in \triangle$.
- If $\tau, \sigma \in \triangle$, then $\tau \cap \sigma$ is a face of both τ and σ.

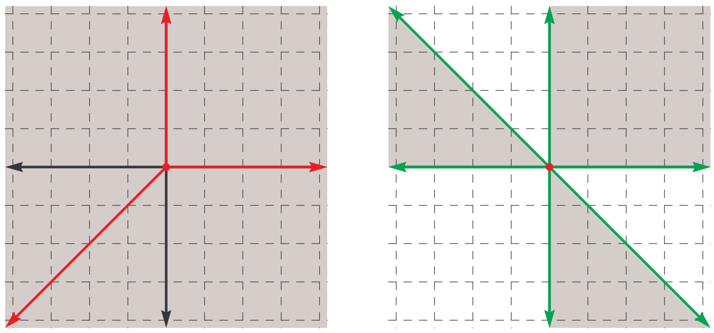

Figure 15: Two fans in the plane.

Lemma 4.7. *If $P \subseteq \mathbb{R}^n$ is an n-dimensional polytope, then the collection of all inner normal cones of the faces of P*

$$\triangle_P = \{ \sigma_F \mid F \in \mathcal{F}(P) \}$$

is a fan. For any polytope P, the fan $\triangle_P$ is called the inner normal fan of P.

Proof. First we will show $\triangle_P$ is a fan by verifying that $\triangle_P$ satisfies the two defining properties of a fan. Let $\sigma_E \in \triangle_P$ with τ a face of σ_E. Lemma 4.3 says

$$\sigma_E = \text{pos}\{a_1, \ldots, a_k\}$$

where $a_1, \ldots, a_k$ are the inner normals to all facets which contain the face E of P. Since τ is a face of σ_E, we can re-index the vectors $a_1, \ldots, a_k$ so that τ is generated by the first j of them. Let $F_1, \ldots, F_j \subseteq P$ be the facets whose inner normals generate τ. By Theorem 2.7, E is the intersection of all facets which contain it. Thus

$$E = \bigcap_{i=1}^{k} F_i \subseteq \bigcap_{i=1}^{j} F_i = G \neq \emptyset.$$

But G is clearly a face of P and $\sigma_G = \text{pos}\{a_1, \ldots, a_j\} = \tau$. Therefore, τ is a face of σ_E implies $\tau \in \triangle_P$.

For the second property, suppose E and H are faces of P so that $\sigma_E, \sigma_H \in \triangle_P$ and let G be the smallest face of P containing both E and H. Lemma 4.5 implies σ_G is a face of both σ_E and σ_H.

Clearly we have $\sigma_G \subseteq \sigma_E \cap \sigma_H$. We claim that in fact, $\sigma_G = \sigma_E \cap \sigma_H$. To see this, consider two cases. In the first place, if E and H are not contained by any common facet, then Lemma 4.3 shows

$$\sigma_E \cap \sigma_H = \text{pos}\{\, a_i \in A \mid (E \cup H) \subseteq F_i \,\} = \{0\}$$

since by Definition 4.1 every inner normal cone contains the origin. Furthermore, since $G = P$ is the smallest face which contains both E and H we have $\sigma_G = \sigma_P = \{0\} = \sigma_E \cap \sigma_H$.

On the other hand, suppose that E and H are contained in at least one common facet. Let $A = \{a_1, \ldots, a_m\}$ be the collection of all inner normal vectors to facets $F_1, \ldots, F_m$ of P. By Lemma 4.3, we have $\sigma_G = \text{pos}\{\, a_i \in A \,|\, G \subseteq F_i \,\}$ and since $(E \cup H) \subseteq G$, we have

$$\begin{aligned}\sigma_E \cap \sigma_H &= \text{pos}\{\, a_i \in A \,|\, (E \cup H) \subseteq F_i \,\} \\ &\subseteq \text{pos}\{\, a_i \in A \,|\, G \subseteq F_i \,\} \\ &= \sigma_G\end{aligned}$$

We have shown that for each pair of cones $\sigma_E, \sigma_H \in \triangle_P$, there is a cone $\sigma_G = (\sigma_E \cap \sigma_H) \in \triangle_P$ such that $\sigma_G \prec \sigma_E$ and $\sigma_G \prec \sigma_H$. Therefore, $\triangle_P$ is a fan. □

Example 4.8. Consider the standard 2-simplex pictured on left hand side of Figure 16.

$$P = \text{conv}\{(0,0),\, (1,0),\, (0,1)\} \subseteq \mathbb{R}^2$$

In the figure, the vertices of P are labeled with lower case letters $a = (0,0)$, $b = (1,0)$ and $c = (0,1)$. For ease of presentation, I have drawn

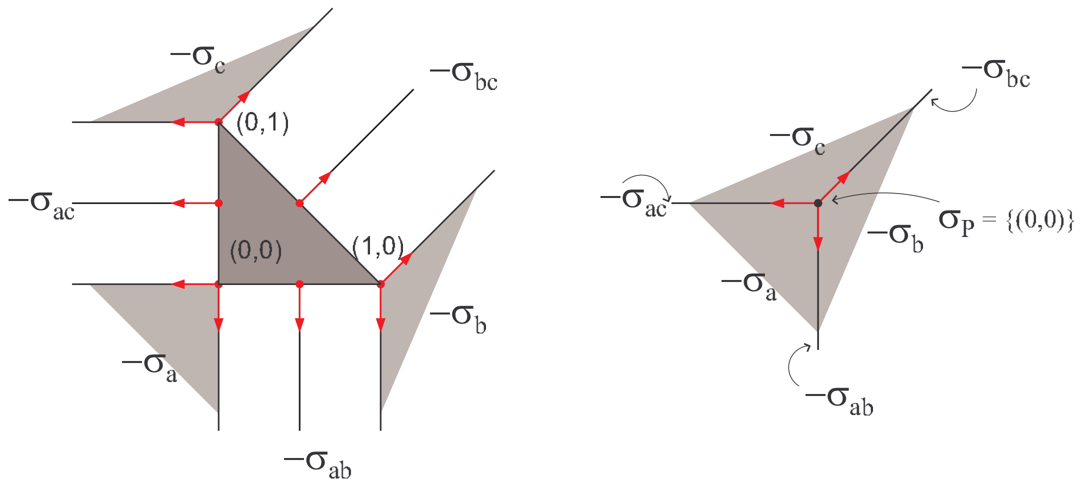

Figure 16: The outward normal fan of the 2-simplex.

the *outward normal fan* of P. That is, the fan

$$\triangle_P^o = \{ -\sigma_F \mid F \in \mathcal{F}(P) \}.$$

Each cone on the right hand side of Figure 16 is the positive hull of the outward pointing normal vectors to facets of P. Since the outward pointing normals differ from the inward pointing normals by a negative constant, we can construct the inner normal fan from the outer normal fan by reflecting through the origin. Figure 17 shows the inner normal fan of the 2-simplex.

Lemma 4.9. *For any polytope P, there exists an inclusion reversing combinatorial isomorphism $\Phi : \mathcal{F}(P) \to \triangle_P$ given by $\Phi(F) = \sigma_F$.*

Proof. By Theorem 2.7 every face of P is the intersection of the facets which contain it. But by Lemma 4.3, the inner normal cone of a face is

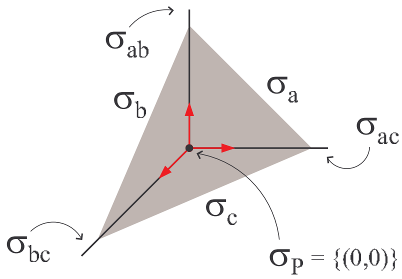

Figure 17: The inner normal fan of the 2-simplex.

the positive hull of the inner normal vectors to the facets containing it. Therefore, if E and F are distinct faces of P, then their inner normal cones are distinct. This implies that $\Phi : \mathcal{F}(P) \to \triangle_P$ is injective.

On the other hand, Φ is clearly surjective since by definition, every cone in $\triangle_P$ is the inner normal cone of some face of P.

Last, let $E,\ F \in \mathcal{F}(P)$. Lemma 4.5 shows that if $E \subseteq F$, then $\sigma_F \subseteq \sigma_E$. □

Smooth Cones and Polytopes

In this section we introduce the concept of smoothness and present several relevant results. The section concludes with the definition of a Selfatope.

Definition 5.1. In the vector space $\mathbb{R}^n$, we refer to points in $\mathbb{Z}^n \subseteq \mathbb{R}^n$ as *lattice points.* A *lattice polytope* is a polytope whose vertices are lattice points.

Example 5.2. The standard 3-dimensional simplex,

$$\triangle_3 = \text{conv}\{\ (0,0,0),\ (1,0,0),\ (0,1,0),\ (0,0,1)\ \} \subseteq \mathbb{R}^3\ ,$$

is a lattice polytope, but the compressed 3-simplex on the right hand side of Figure 18 is not.

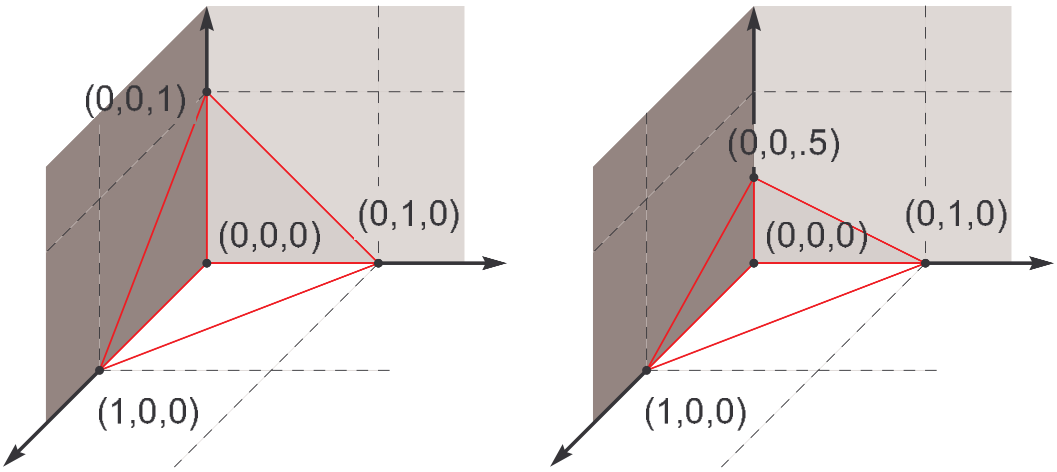

Figure 18: Lattice simplex

Definition 5.3. Let $\sigma \subseteq \mathbb{R}^n$ be a rational convex polyhedral cone. We say that σ is a *smooth cone* if a minimal generating set of σ can be extended to a form basis for $\mathbb{Z}^n$.

Let p be a vertex of the lattice polytope $P \subseteq \mathbb{R}^n$. For each edge $E_i \subseteq P$ which contains p, define w_i to be the closest lattice point to p in the edge E_i. We say that the polytope P is *smooth at the vertex* p if the set

$$\{ (w_1 - p), \ldots, (w_k - p) \}$$

can be extended to a basis for $\mathbb{Z}^n$. We refer to the vectors $(w_i - p)$ as *edge vectors* at p. If P is smooth at each of its vertices, then we say P is a *smooth* lattice polytope.

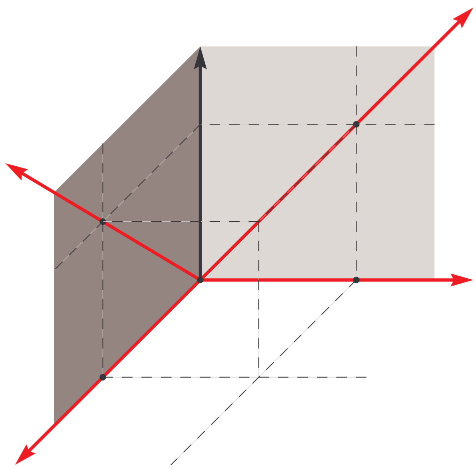

Figure 19: A rational 3-dimensional cone that is not smooth.

Example 5.4. Consider the two cones

$$\sigma = \text{pos}\{(1,0,0),\ (0,1,0)\}$$

$$\text{and} \qquad \tau = \text{pos}\{(1,0,0),\ (0,1,0),\ (1,0,1),\ (0,1,1)\}.$$

Both cones are subsets of $\mathbb{R}^3$, but only σ is smooth. This is clear because the σ is generated by a subset of the standard basis on $\mathbb{R}^n$. In contrast, τ cannot be generated by fewer than four vectors. Therefore τ is not smooth since every generating set is linearly dependent.

It is not always easy to determine if a cone or polytope is smooth by directly applying the definition. However, we do have a powerful tool for testing the smoothness of polytopes and cones.

Lemma 5.5. *Suppose that $\{a_1, \ldots, a_n\} \subseteq \mathbb{Z}^n$ is a minimal primitive generating set for the rational n-dimensional convex polyhedral cone $\sigma \subseteq \mathbb{R}^n$. The cone σ is smooth if and only if* $\det A = \det A^{-1} = \pm 1$ *where*

$$A = \begin{pmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{pmatrix}$$

Proof. If $\sigma \subseteq \mathbb{R}^n$ is smooth, then the generating set $\{a_1, \ldots, a_n\}$ forms a basis for $\mathbb{Z}^n$. Therefore the matrix A represents a change of basis $A : \mathbb{R}^n \to \mathbb{R}^n$ from the standard basis $\{e_1, \ldots, e_n\}$ to the new basis

$\{a_1, \ldots, a_n\}$. This means that A is an invertible matrix and

$$1 = \det I = \det(AA^{-1}) = \det A \det A^{-1}.$$

However, $\{a_1, \ldots, a_n\} \subseteq \mathbb{Z}^n$ by the definition of a smooth cone and hence $\det A \in \mathbb{Z}$ and $\det A^{-1} \in \mathbb{Z}$. Therefore, $\det A = \det A^{-1} = \pm 1$.

On the other hand, suppose $\det A = \det A^{-1} = \pm 1$. Then the generating set $\{a_1, \ldots, a_n\} \subseteq \mathbb{Z}^n$ is clearly linearly independent. To see that they span $\mathbb{Z}^n$, observe that because A and A^{-1} are invertible, they represent a bijective functions $\mathbb{Z}^n \to \mathbb{Z}^n$. Thus if $z \in \mathbb{Z}^n$, then $A^{-1}z$ is a representation of z as a $\mathbb{Z}$-linear combination of the vectors $\{a_1, \ldots, a_n\}$. □

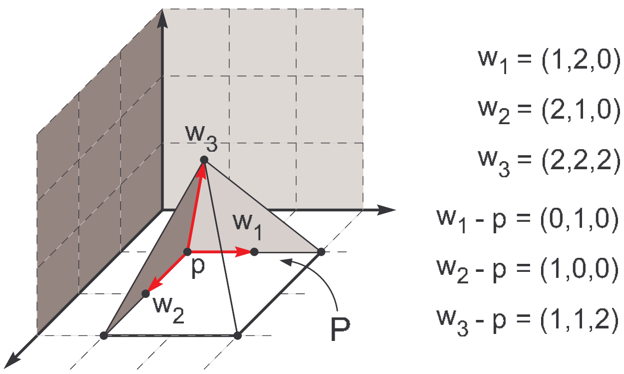

Figure 20:

Example 5.6. Consider the 3-dimensional polytope P in Figure 20 above.

$$P = \operatorname{conv}\{\, (1,1,0),\ (3,1,0),\ (3,3,0),\ (1,3,0),\ (2,2,2)\,\}$$

We would like to check to see if this polytope is smooth at the vertex $p = (1, 1, 0) \in P$. Of the three edges which contain the vertex p, only one is lattice free. Along the other two edges, the closest lattice points to p are $w_1 = (1, 2, 0)$ and $w_2 = (2, 1, 0)$.

By definition, P is smooth at the vertex p if the vectors pictured in red form a basis for $\mathbb{Z}^3$. They do not, as is easily verified by computing the determinant.

$$\det \begin{pmatrix} - & (w_1 - p) & - \\ - & (w_2 - p) & - \\ - & (w_3 - p) & - \end{pmatrix} = \det \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 2 \end{pmatrix} = -2 \neq \pm 1$$

Definition 5.7. A linear transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ is called *unimodular* if the restriction to $\mathbb{Z}^n \to \mathbb{Z}^n$ is bijective. Equivalently, a transformation is unimodular if and only if $A \in GL_n(\mathbb{Z})$ and $\det A = \pm 1$.

Lemma 5.8. *A rational n-dimensional convex polyhedral cone $\sigma \subseteq \mathbb{R}^n$ is smooth if and only if there exists a unimodular transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ such that $A\sigma = \mathrm{pos}\{ e_1, \ldots, e_n \}$, where the e_i represent the standard basis vectors in $\mathbb{R}^n$.*

Proof. Suppose that $\sigma = \mathrm{pos}\{a_1, \ldots, a_n\} \subseteq \mathbb{R}^n$ is smooth. Then the generating set forms a basis for $\mathbb{Z}^n$ and therefore there exists a change of basis $A : \mathbb{Z}^n \to \mathbb{Z}^n$ such that $Aa_i = e_i$ for all $1 \leq i \leq n$. Of course changes of basis are invertible, so $A^{-1} : \mathbb{Z}^n \to \mathbb{Z}^n$ exists, and therefore

A must be a bijective function on $\mathbb{Z}^n$. By extending A linearly, we get a unimodular transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\begin{aligned} A\sigma &= A(\text{pos}\{a_1, \ldots, a_n\}) \\ &= \text{pos}\{Aa_1, \ldots, Aa_n\} \\ &= \text{pos}\{e_1, \ldots, e_n\} \end{aligned}$$

Now suppose that $\sigma \subseteq \mathbb{R}^n$ is a rational n-dimensional convex polyhedral cone and let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a unimodular transformation such that $A\sigma = \text{pos}\{\, e_1, \ldots, e_n \,\}$. Then the inverse transformation A^{-1} exists and

$$\sigma = A^{-1}(\text{pos}\{e_1, \ldots, e_n\}) = \text{pos}\{A^{-1}e_1, \ldots, A^{-1}e_n\}.$$

Now we compute

$$\det \begin{pmatrix} | & & | \\ A^{-1}e_1 & \ldots & A^{-1}e_n \\ | & & | \end{pmatrix} = \det A^{-1}I = \det A^{-1} = \pm 1.$$

Therefore, σ is smooth by Lemma 5.5. □

Lemma 5.9. *If $\sigma \subseteq \mathbb{R}^n$ is a smooth n-dimensional cone, then σ has a minimal generating set consisting of precisely n vectors.*

Proof. Suppose that $\sigma = \text{pos}\{\, a_1, \ldots, a_k \,\} \subseteq \mathbb{R}^n$ is a smooth n-dimensional cone. Since σ is n-dimensional, we must have

$$\operatorname{span}(\sigma) = \operatorname{span}\{\, a_1, \, \ldots, \, a_k \,\} = \mathbb{R}^n.$$

Hence $k \geq n$. But because σ is smooth, the minimal generating set $\{\, a_1, \, \ldots, \, a_k \,\}$ can be extended to form a basis for $\mathbb{Z}^n$ and therefore must be linearly independent. This implies $k \leq n$. □

Lemma 5.10. *If $\sigma \subseteq \mathbb{R}^n$ is a smooth n-dimensional cone with minimal generating set $\mathcal{A} = \{\, a_1, \, \ldots, \, a_n \,\}$, then σ has precisely n facets, and all are of the form*

$$F_j = \operatorname{pos}\{\, a_1, \, \ldots, \, a_{j-1}, \, a_{j+1}, \, \ldots, \, a_n \,\}$$

Proof. First observe that every face of $\sigma = \operatorname{pos} \mathcal{A}$ is generated by some subset of $\mathcal{A}$. But a facet must have dimension $n-1$ and therefore a facet must be generated by a subset with at least $n-1$ elements.

Second, note that because $\sigma \subseteq \mathbb{R}^n$ is smooth, the generating set $\{\, a_1, \, \ldots, \, a_n \,\}$ forms a basis for the vector space $\mathbb{R}^n$ and hence each $n-1$ element subset of is linearly independent. For each $1 \leq j \leq n$ we have

$$\begin{aligned} & 0 \, \in \, \operatorname{span}\{\, a_1, \, \ldots, \, a_{j-1}, \, a_{j+1}, \, \ldots, \, a_n \,\} \\ \text{and} \qquad & \dim(\operatorname{span}\{\, a_1, \, \ldots, \, a_{j-1}, \, a_{j+1}, \, \ldots, \, a_n \,\}) = n-1. \end{aligned}$$

This shows that the span of each $n-1$ element subset is an affine hyperplane which contains the origin. Therefore, there exists a non-zero vector $b_j \in \mathbb{R}^n$ such that

$$\begin{aligned} H_j &= \{\, x \in \mathbb{R}^n \mid \langle b_j, x \rangle = 0 \,\} \\ &= \operatorname{span}\{\, a_1,\, \ldots,\, a_{j-1},\, a_{j+1},\, \ldots,\, a_n \,\}. \end{aligned}$$

for each $1 \le j \le n$. Choose the orientation of b_j so that $\langle b_j, a_j \rangle \ge 0$.

We claim that H_j is a supporting hyperplane for a facet of σ. To see this, suppose that $x \in \sigma$. Then there are positive real numbers $\lambda_1, \ldots, \lambda_n$ such that

$$x = \lambda_1 a_1 + \cdots + \lambda_n a_n$$

We have

$$\begin{aligned} \langle b_j, x \rangle &= \langle b_j, \lambda_1 a_1 + \cdots + \lambda_n a_n \rangle \\ &= \lambda_1 \langle b_j, a_1 \rangle + \ldots \lambda_n \langle b_j, a_n \rangle \\ &= \lambda_j \langle b_j, a_j \rangle \\ &\ge 0. \end{aligned}$$

Hence $\sigma \subseteq H_j^+$. Finally, observe

$$F_j = H_j \cap \sigma = \operatorname{pos}\{\, a_1,\, \ldots,\, a_{j-1},\, a_{j+1},\, \ldots,\, a_n \,\} \neq \emptyset$$

and clearly $\dim(F_j) = n - 1$. Therefore, F_j is a facet of σ.

We have shown that every facet of σ is generated by an $n-1$ elements subset of $\{ a_1, \ldots, a_n \}$ and that every $n-1$ element subset generates a facet. But there are obviously only n such subsets and so σ has precisely n facets. □

Lemma 5.11. *If σ is smooth cone, then $\sigma^\vee$ is smooth.*

Proof. Suppose that the set $\{ a_1, \ldots, a_n \} \subseteq \mathbb{R}^n$ forms a basis for $\mathbb{R}^n$ so that the cone

$$\sigma = \text{pos}\{ a_1, \ldots, a_n \}$$

is smooth and n-dimensional. Then there exists a unimodular change of basis $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $Aa_i = e_i$ for all $1 \leq i \leq n$. By Lemma 5.8, σ is unimodularly equivalent to the first orthant.

$$A\sigma = \text{pos}\{ e_1, \ldots, e_n \} := \tau.$$

Now we compute the dual cone.

$$\begin{aligned}
\sigma^\vee &= \{ x \in \mathbb{R}^n \mid \langle s, x \rangle \geq 0, \ \forall (s \in \sigma) \} \\
&= \{ x \in \mathbb{R}^n \mid \langle At, x \rangle \geq 0, \ \forall (t \in \tau) \} \\
&= \{ x \in \mathbb{R}^n \mid \langle t, A^T x \rangle \geq 0, \ \forall (t \in \tau) \} \\
&= \{ (A^T)^{-1} y \in \mathbb{R}^n \mid \langle t, y \rangle \geq 0, \ \forall (t \in \tau) \} \\
&= (A^T)^{-1} \tau^\vee = (A^T)^{-1} \tau
\end{aligned}$$

Therefore, $A^T\sigma^\vee = \tau$. This implies that $\sigma^\vee$ is unimodularly equivalent to the smooth cone τ and hence $\sigma^\vee$ is smooth. $\square$

Definition 5.12. Suppose that $P \subseteq \mathbb{R}^n$ is a lattice polytope and let $E = \text{conv}\{p, q\}$ be an edge. If $E \cap \mathbb{Z}^n = \{p, q\}$, then we say that the edge E is *lattice free*. A *selfatope* is a smooth lattice polytope with lattice free edges.

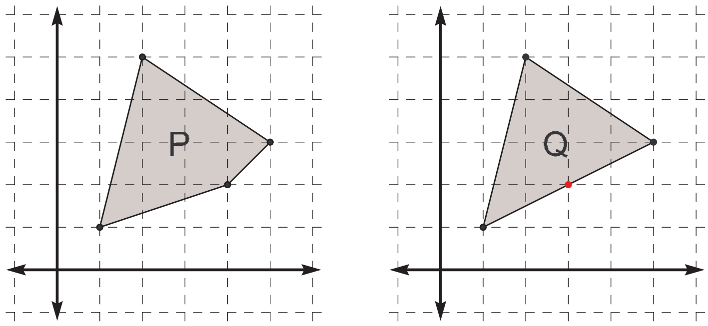

Figure 21: P has lattice free edges but Q does not.

Example 5.13. Figure 21 shows two planar polytopes P and Q. The polytope P has lattice free edges, while Q has an edge which is not lattice free.

Neither P nor Q is a selfatope because neither is smooth. However, there are many examples of selfatopes. The most simple examples are the n-simplex and the n-cube.

$$\triangle_n = \text{conv}\{0, e_1 ..., e_n\} \subseteq \mathbb{R}^n$$

$$C_n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \,|\, 0 \leq x_i \leq 1, \ \forall (1 \leq i \leq n)\}$$

Both $\triangle_n$ and C_n are clearly lattice polytopes and it is easy to verify that both are smooth and lattice free.

Definition 5.14. A fan $\triangle$ is called *smooth* if σ is a smooth cone for each $\sigma \in \triangle$.

Lemma 5.15. *A polytope is smooth if and only if its inner normal fan $\triangle_P$ is smooth.*

Proof. Suppose that P is a smooth polytope. To see that $\triangle_P$ is a smooth fan, it is enough to show that for each vertex $p \in P$, the cone $\sigma_p \in \triangle_P$ is smooth. This is because for any face $F \succ p$, Lemma 4.5 gives us $\sigma_F \prec \sigma_p$. This implies that the generators of σ_F are a subset of the generators of σ_p. Therefore, if the generators of σ_p can be extended to a basis for $\mathbb{Z}^n$, so can the generators of σ_F.

Let $p \in \operatorname{vert} P$ and suppose that $(w_1 - p), \ldots, (w_n - p)$ are the edge vectors emanating from p. Since P is a smooth polytope, it is clear that the cone

$$\tau_p = \operatorname{pos}\{\, (w_1 - p), \ldots, (w_n - p) \,\}$$

is smooth. However, Lemma 4.4 says that $\sigma_p^\vee = \tau_p$ and thus Lemma 5.11 shows that σ_p is smooth. Hence, if P is a smooth polytope, then its inner normal fan $\triangle_P$ is smooth.

Now let $\triangle_P$ be the inner normal fan of the polytope P and suppose that $\triangle_P$ is smooth. For each vertex $p \in P$ there is a cone $\sigma_p \in \triangle_P$.

Furthermore, σ_p is smooth because the inner normal fan $\triangle_P$ is smooth. By Lemma 4.4, we have $\sigma_p^\vee = \tau_p$. From here, Lemma 5.11 implies that τ_p is smooth. □

Fans of Selfatopes

We now have enough tools to precisely state the main theorem.

Main Theorem. *If P and Q are selfatopes with $\triangle_P = \triangle_Q$, then there exists a vector r such that $Q = r + P$.*

The proof relies upon ideas from all of the previous sections, but is primarily combinatorial in nature. However, before proceeding to the proof, it is instructive to see that the statement of the main theorem is false for general polytopes.

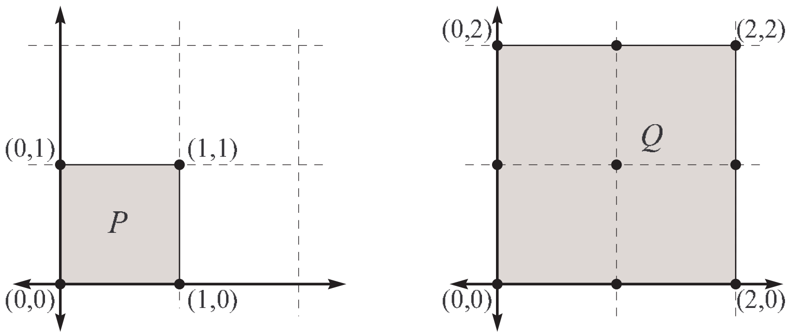

Figure 22: Different smooth lattice polytopes with the same inner normal fan.

Example 6.1. In the first place, consider what happens if the requirement for lattice free edges is dropped. The polytopes P and Q defined below and pictured in Figure 22 are both smooth lattice polytopes with the same inner normal fan.

$$P = \text{conv}\{(0,0),\ (1,0),\ (1,1),\ (0,1)\}$$

$$Q = \text{conv}\{(0,0),\ (2,0),\ (2,2),\ (0,2)\} = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right) P$$

Notice that the inner normal fans of P and Q are the same, but Q is not a translation of P. However, the main theorem is not contradicted because Q does not have lattice free edges and therefore it is not a selfatope.

On the other hand, if we drop only the lattice polytope requirement, then a similar situation occurs. For example consider P as above, and the polytope R in Figure 23.

$$R = \text{conv}\{(0,0),\ (\tfrac{1}{2},0),\ (\tfrac{1}{2},\tfrac{1}{2}),\ (0,\tfrac{1}{2})\} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} P$$

In this situation, P and R have lattice free edges and they have the same inner normal fan. But again, R is not a translation of P. We must conclude that both the lattice polytope and the lattice free edge conditions are required for the main theorem to hold.

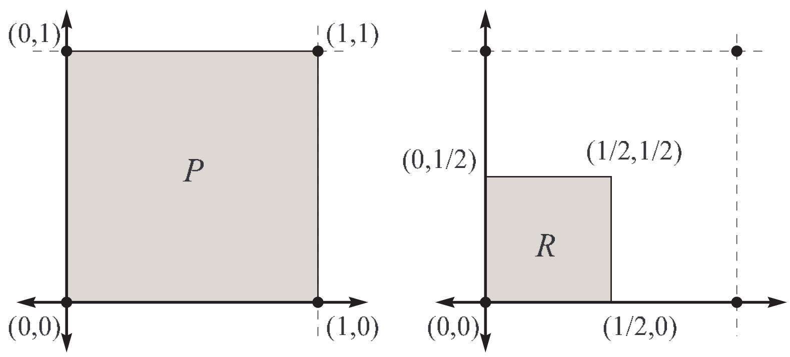

Figure 23: Different polytopes with the same inner normal fan.

In the proof of the main theorem, we use certain combinatorial properties of smooth polytopes. It is for this reason that the main theorem requires P and Q to be smooth. However, since we have not been able to construct a counter-example to show that smoothness is absolutely required, we present the following as a possible topic for further research.

Conjecture. *If P and Q are lattice polytopes with lattice free edges and $\triangle_P = \triangle_Q$, then there exists a vector r such that $Q = r + P$.*

The remainder of this section is divided into three parts. In the first part, we prove some facts about the combinatorial structure of smooth polytopes. In the second part, we will present two consequences for general polytopes that have the same inner normal fan. Once these tasks are complete, we will proceed to the proof of the main theorem.

Lemma 6.2. *If $P \subseteq \mathbb{R}^n$ is a smooth full dimensional lattice polytope, then every vertex of P is adjacent to precisely n vertices of P.*

Proof. Suppose that $P \subseteq \mathbb{R}^n$ is a smooth full dimensional lattice polytope and let $p \in P$ be any vertex. From Lemma 2.6 we know that there are at least n vertices of P which are adjacent to p. If the set of vertices adjacent to p are

$$\{ w_1, \dots, w_k \},$$

then $k \leq n$ so that the vectors pointing in the direction of the edges of p are

$$\{ (w_1 - p), \dots, (w_k - p) \}$$

Recall that since P is a smooth lattice polytope, there are positive rational numbers $r_1, \ldots, r_k$ so that the set

$$\{ r_1(w_1 - p), \ldots, r_k(w_k - p) \}$$

can be extended to a basis for $\mathbb{Z}^n$. Therefore this is a linearly independent set. But $k \geq n$ and clearly each of these vectors is an element of $\mathbb{Z}^n$, so we must have $k = n$. Therefore, every vertex of P is adjacent to precisely n vertices of P. □

Corollary 6.3. *If $P \subseteq \mathbb{R}^n$ is a smooth n-dimensional lattice polytope, then every vertex $p \in P$ is contained by precisely n facets.*

Proof. Let $p \in P$ be a vertex of the smooth full dimensional polytope $P \in \mathbb{R}^n$. By Lemma 6.2 above, there are precisely n vertices of P adjacent to p. Denote these vertices $w_1, \ldots, w_n$. Lemma 4.4 shows that

$$\sigma_p^\vee = \text{pos}\{ (w_1 - p), \ldots, (w_n - p) \}$$

where σ_p denotes the inner normal cone of the face p. Since P is smooth, σ_p is smooth by Lemma 5.15.

Now let $a_1, \ldots, a_k$ be the inner normal vectors to the facets containing p. By Lemma 4.3, we have

$$\sigma_p = \text{pos}\{a_1, \ldots, a_k\}.$$

But Lemma 5.9 says that since σ_p is smooth, it has precisely n primitive generators. Therefore $k = n$ and p is contained in exactly n facets of P. □

Corollary 6.4. *Let $P \subseteq \mathbb{R}^n$ be a smooth n-dimensional lattice polytope. If $E \subseteq P$ is an edge, then E is contained in precisely $n-1$ facets.*

Proof. Suppose that $E \subseteq P$ is an edge. Then there are distinct vertices p, $q \in P$ such that $E = \text{conv}\{p, q\}$. Certainly any facet which contains E must also contain both p and q. However, Corollary 6.3 shows that p and q are each contained by n facets of P. But the facets which contain p cannot coincide with the facets containing q or else we would have $p = q$. Therefore, E can be contained in at most $n-1$ facets.

On the other hand, suppose that E is contained in exactly k facets. We claim that the inner normal vectors $a_1, \ldots, a_k$ to these facets form a linearly independent set. This is clear, because the vectors $a_1, \ldots, a_k$ each generate a face of the smooth inner normal cones σ_p and σ_q. Now we can apply Lemma 1.14 and Lemma 1.15 to see that

$$1 = \dim E = \dim(\ker A) = n - k.$$

Therefore, $k = n - 1$. □

Since every edge of a polytope is the convex hull of two adjacent vertices, Corollary 6.4 is equivalent to Corollary 6.5 below.

Corollary 6.5. *If $P \subseteq \mathbb{R}^n$ is a smooth n-dimensional polytope and p, $q \in P$ are adjacent vertices, then p and q are contained in exactly $(n-1)$ common facets of P.*

Now that we have established some combinatorial characteristics of smooth polytopes, we will move on to consider what it means for two general polytopes to share the same inner normal fan.

Lemma 6.6. *If P and Q are polytopes and with the same inner normal fan $\triangle_P = \triangle_Q$, then there exists a combinatorial isomorphism $\varphi : \mathcal{F}(P) \longrightarrow \mathcal{F}(Q)$ with the property*

$$\sigma_F = \sigma_{\varphi(F)}\,, \qquad \forall\, F \in \mathcal{F}(P)$$

where σ_F is the cone associated to the face F.

Proof. Suppose that P and Q are polytopes with $\triangle_P = \triangle_Q$. There is an obvious one-to-one correspondence between the fans $\triangle_P = \triangle_Q$ given by the identity map. By Lemma 4.9, there exists a bijection $\mathcal{F}(P) \to \triangle_P$. Therefore, we have a composition of bijections

$$\mathcal{F}(P) \xrightarrow{\ \alpha\ } \triangle_P = \triangle_Q \xrightarrow{\ \beta\ } \mathcal{F}(Q)$$

where $\alpha(F) = \sigma_F$, and $\beta(\sigma_G) = G$ are the bijections from Lemma 4.9.

If we let $\varphi = (\beta \circ \alpha)$, then clearly

$$\varphi : \mathcal{F}(P) \to \mathcal{F}(Q)$$

is a bijection.

Next, observe that for every face $F \in \mathcal{F}(P)$ we have

$$\begin{aligned} \sigma_F &= \alpha(F) \\ &= \beta^{-1} \circ (\beta \circ \alpha)(F) \\ &= \beta^{-1}(\varphi(F)) \\ &= \sigma_{\varphi(F)}. \end{aligned}$$

Finally, we can apply Lemma 4.7 to see that for each pair of faces F_1, $F_2 \in \mathcal{F}(P)$, we have

$$\begin{aligned} F_1 \subseteq F_2 \quad &\Longleftrightarrow \quad \sigma_{F_2} \subseteq \sigma_{F_1} \\ &\Longleftrightarrow \quad \sigma_{\varphi(F_2)} \subseteq \sigma_{\varphi(F_1)} \\ &\Longleftrightarrow \quad \varphi(F_1) \subseteq \varphi(F_2) \end{aligned}$$

Therefore $\varphi : \mathcal{F}(P) \to \mathcal{F}(Q)$ is a combinatorial isomorphism with the property $\sigma_F = \sigma_{\varphi(F)}$ for every face $F \in \mathcal{F}(P)$. □

Lemma 6.7. *Let P and Q be full dimensional polytopes expressed in polyhedral form where the rows of A (respectively B) are inward pointing* *normal vectors to the facets of P (respectively Q). For example,*

$$P = \{ x \in \mathbb{R}^n \mid Ax \geq \alpha \}$$
$$\textit{and} \quad Q = \{ x \in \mathbb{R}^n \mid Bx \geq \beta \}$$

If $\triangle_P = \triangle_Q$, then $B = \Lambda\Phi A$, where Φ is a permutation of the row vectors and Λ is a non-singular real diagonal matrix.

Proof. First observe that the matrices A and B have the same number of columns because P and Q are both n dimensional. Second, Lemma 6.6 shows that P and Q are combinatorially isomorphic. Therefore, the facets of P are in one to one correspondence with the facets of Q. Consequently, A and B have the same number of rows.

Since $\triangle_P = \triangle_Q$, Lemma 6.6 says that there exists a combinatorial isomorphism $\varphi : \mathcal{F}(P) \to \mathcal{F}(Q)$. with the property $\sigma_F = \sigma_{\varphi(F)}$. Let F be a facet of P so that $\varphi(F)$ is a facet of Q. Lemma 3.3 shows us that the cone σ_F is generated by a normal vector to the facet F. Therefore, F and $\varphi(F)$ have normal vectors which only differ by a positive real constant.

Now recall that the rows of A are the normal vectors to the facets of P and similarly, the rows of B are the normal vectors to the facets of Q. We have shown that $A = B$ up to a possible re-ordering and re-scaling of the rows. Hence $B = \Lambda\Phi A$, where Φ is a permutation of the row vectors and Λ is a non-singular real diagonal matrix. $\square$

Finally we are ready to present the main result. We have broken the main theorem into two parts. The proof of Theorem 6.8 lays the ground work for the main theorem which follows Theorem 6.8 as a corollary.

Theorem 6.8. *Let $P \subseteq \mathbb{R}^n$ and $Q \subseteq \mathbb{R}^n$ be n-dimensional selfatopes with the same inner normal fan $\triangle_P = \triangle_Q$. If $\varphi : \mathcal{F}(P) \to \mathcal{F}(Q)$ is a combinatorial isomorphism with the property $\sigma_F = \sigma_{\varphi(F)}$ for every face $F \in \mathcal{F}(P)$, then*

$$x - y = \varphi(x) - \varphi(y)$$

for every pair of adjacent vertices $x, y \in \operatorname{vert} P$.

Proof. Let P and Q be full dimensional selfatopes such that $\triangle_P = \triangle_Q$ and let $\varphi : \mathcal{F}(P) \longrightarrow \mathcal{F}(Q)$ be a combinatorial isomorphism such that $\sigma_F = \sigma_{\varphi(F)}$ for all $F \in \mathcal{F}(P)$.

Suppose that x and y are adjacent vertices in P. By Corollary 6.5 x and y are contained in exactly $n - 1$ common facets of P. Similarly, $\varphi(x)$ and $\varphi(y)$ are contained in exactly $n - 1$ common facets of Q because φ is a combinatorial isomorphism. Let

$$\{F_1, \ldots, F_{n-1}\} = \{\, F \in \operatorname{facet} P \mid x, y \in F \,\}$$

Clearly,

$$x,\, y \in \bigcap_{i=1}^{n-1} F_i \quad \Longrightarrow \quad \varphi(x),\, \varphi(y) \in \bigcap_{i=1}^{n-1} \varphi(F_i)$$

because φ has the property $\sigma_F = \sigma_{\varphi(F)}$ for every face $F \in \mathcal{F}(P)$. Furthermore, Lemma 6.7 shows that for each i, the facets $F_i \subseteq P$ and $\varphi(F_i) \subseteq Q$ have the same inward pointing normal vectors. Therefore, the supporting hyperplanes of F_i and $\varphi(F_i)$, respectively have the form

$$\begin{aligned} F_i \subseteq H_i &= \{\, x \in \mathbb{R}^n \mid \langle a_i, x \rangle = \alpha_i \,\} \\ \varphi(F_i) \subseteq \widetilde{H_i} &= \{\, x \in \mathbb{R}^n \mid \langle a_i, x \rangle = \beta_i \,\}. \end{aligned}$$

In particular, we can write

$$\begin{aligned} &\bigcap_{i=1}^{n-1} F_i \qquad \subseteq \bigcap_{i=1}^{n-1} H_i \;=\; \{\, x \in \mathbb{R}^n \mid Ax = \alpha \,\} \\ \text{and} \quad &\bigcap_{i=1}^{n-1} \varphi(F_i) \subseteq \bigcap_{i=1}^{n-1} \widetilde{H_i} \;=\; \{\, x \in \mathbb{R}^n \mid Ax = \beta \,\}. \end{aligned}$$

where A is the $(n-1) \times n$ matrix

$$A = \begin{pmatrix} - & a_1 & - \\ & \vdots & \\ - & a_{n-1} & - \end{pmatrix}.$$

Specifically, the rows of A are the inward pointing normal vectors to the $(n-1)$ facets which contain both x and y.

We claim that the row vectors of A are linearly independent. To see this, observe that $E = \operatorname{conv}\{x, y\} \in \mathcal{F}(P)$ is an edge of P and therefore

$\sigma_E \in \triangle_P$. Since $\triangle_P$ is a smooth fan, σ_E is clearly a smooth cone and consequently its generating vectors are linearly independent. However, Lemma 4.3 shows that $\sigma_E = \text{pos}\{a_1, \ldots, a_{n-1}\}$, and hence the rows of A are independent.

Now we can apply Lemma 1.14 to arrive at

$$\begin{aligned} x \in \bigcap_{i=1}^{n-1} F_i \quad &\subseteq \bigcap_{i=1}^{n-1} H_i \;=\; y + \ker A \\ \text{and} \qquad \varphi(x) \in \bigcap_{i=1}^{n-1} \varphi(F_i) &\subseteq \bigcap_{i=1}^{n-1} \widetilde{H_i} \;=\; \varphi(y) + \ker A. \end{aligned}$$

Certainly, $x \in y + \ker A$ implies $(x - y) \in \ker A$ and similarly, $\varphi(x) \in \varphi(y) + \ker A$ implies $(\varphi(x) - \varphi(y)) \in \ker A$. By the rank nullity theorem, $\ker A$ is 1-dimensional. Hence,

$$\ker A = \text{span}\{\varphi(x) - \varphi(y)\} = \text{span}\{x - y\}.$$

Therefore, there exists a non-zero real number λ such that $\varphi(x) - \varphi(y) = \lambda(x - y)$. Since P and Q are selfatopes, $\text{vert}\, P \subseteq \mathbb{Z}^n$ and $\text{vert}\, Q \subseteq \mathbb{Z}^n$. Hence, λ is actually a rational number.

We claim that in fact, $\lambda = 1$. To see this, we will begin by showing that $\lambda = \pm 1$. There are four cases. In the first place if $\lambda \in (0, 1]$, then

$$\begin{aligned} z &= \lambda x + (1 - \lambda) y \\ &= \lambda(x - y) + y \\ &= \varphi(x) - \varphi(y) + y \in \mathbb{Z}^n \end{aligned}$$

Clearly, $z \in \mathbb{Z}^n$ because it is the sum of lattice points. However, if $z \in \text{conv}\{x, y\} = E$ with $z \neq x$ and $z \neq y$, then the edge E is not lattice free. This is a contradiction since P is a selfatope. Therefore, we must have $\lambda = 1$.

On the other hand if $\lambda \in [-1, 0)$ then

$$\begin{aligned} z &= -\lambda x + (1 + \lambda) y \\ &= -\lambda(x - y) + y \\ &= \varphi(y) - \varphi(x) + y \in \mathbb{Z}^n \end{aligned}$$

which implies $\lambda = -1$.

Now suppose that $\lambda > 1$. In this case we have $\frac{1}{\lambda} \in (0, 1)$ and

$$\begin{aligned} x - y &= \tfrac{1}{\lambda}(\varphi(x) - \varphi(y)) \\ x - y + \varphi(y) &= \tfrac{1}{\lambda}(\varphi(x) - \varphi(y)) + \varphi(y) \end{aligned}$$

$$x - y + \varphi(y) = \tfrac{1}{\lambda}\varphi(x) + (1 - \tfrac{1}{\lambda})\varphi(y). \tag{1}$$

The left hand side of equation 1 is a lattice point, but the right side is an interior point of the edge $\operatorname{conv}\{\varphi(x), \varphi(y)\} \subseteq Q$. Once again, this is a contradiction because Q is a selfatope. A similar argument shows that $\lambda < -1$ is impossible.

Now we show that in fact $\lambda = 1$. Suppose that $\lambda = -1$. Then for every $a \in \mathbb{R}^n$, we would have

$$
\begin{aligned}
x - y &= \varphi(y) - \varphi(x) \\
\langle a,\, x - y\rangle &= \langle a,\, \varphi(y) - \varphi(x)\rangle \\
\langle a,\, x\rangle - \langle a,\, y\rangle &= \langle a,\, \varphi(y)\rangle - \langle a,\, \varphi(x)\rangle \\
\langle a,\, x\rangle + \langle a,\, \varphi(x)\rangle &= \langle a,\, y\rangle + \langle a,\, \varphi(y)\rangle \qquad (2)
\end{aligned}
$$

But since $\operatorname{conv}\{x, y\} \subseteq P$ is an edge, Corollary 6.4 and Corollary 6.3 show that there is a unique facet $F_y \in \mathcal{F}(P)$ which contains the vertex y but does not contain x. Let

$$F_y \subseteq H_y = \{\, r \in \mathbb{R}^n \mid \langle a_y,\, r\rangle = 1 \,\}$$

be a supporting hyperplane for F_y so that P is contained in the half-space H_y^+. In particular, we must have

$$\langle a_y,\, x\rangle > \langle a_y,\, y\rangle \quad \text{and} \quad \langle a_y,\, \varphi(x)\rangle > \langle a_y,\, \varphi(y)\rangle.$$

Then,

$$\langle a_y, x\rangle + \langle a_y, \varphi(x)\rangle > \langle a_y, y\rangle + \langle a_y, \varphi(y)\rangle$$

which contradicts equation (2). Hence, $\lambda = -1$ is impossible. We must conclude

$$x - y = \varphi(x) - \varphi(y)$$

for every pair of adjacent vertices $x, y \in P$. $\square$

Now the main theorem follows as a corollary.

Corollary 6.9 (Main Theorem)**.** *If P and Q are selfatopes with $\triangle_P = \triangle_Q$, then there exists a vector r such that $Q = r + P$.*

Proof. Let P and Q be selfatopes with $\triangle_P = \triangle_Q$ and suppose that

$$\varphi : \mathcal{F}(P) \to \mathcal{F}(Q)$$

is a combinatorial isomorphism with the property $\sigma_F = \sigma_{\varphi(F)}$ for all $F \in \mathcal{F}(P)$.

Now recall Balinski's Theorem [5][page 95], which states that the edge vertex graph of a polytope is connected and thus path-connected. Therefore, given any pair of vertices $p, q \in \mathrm{vert}(P)$, there is a path in the edge-vertex graph of P from p to q. Suppose we are given some such path. Then there is a finite sequence of vertices

$$v_1\,(= p),\ \ldots,\ v_i,\ v_{i+1},\ \ldots,\ v_k\,(= q)$$

such that v_i is adjacent to v_{i+1} for all $i \leq k-1$. Because φ is a combinatorial isomorphism, this implies that

$$\varphi(v_1),\ \ldots,\ \varphi(v_i),\ \varphi(v_{i+1}),\ \ldots,\ \varphi(v_k)$$

is a finite sequence of vertices in Q such that $\varphi(v_i)$ is adjacent to $\varphi(v_{i+1})$ for all $i \leq k-1$.

Now apply Lemma 6.8 to the sum

$$\begin{aligned} \sum_{i=1}^{k-1}(v_i - v_{i+1}) &= \sum_{i=1}^{k-1}(\varphi(v_i) - \varphi(v_{i+1})) \\ v_1 - v_k &= \varphi(v_1) - \varphi(v_k) \\ p - q &= \varphi(p) - \varphi(q) \end{aligned}$$

Therefore,

$$p - q = \varphi(p) - \varphi(q)$$

for every pair of vertices $p,\ q \in \text{vert}(P)$. In particular, we can fix a vertex $q_0 \in P$ and write

$$\begin{aligned} \varphi(p) &= \varphi(q_0) + (p - q_0) \\ &= (\varphi(q_0) - q_0) + p \\ &= r + p \end{aligned}$$

where $r = \varphi(q_0) - q_0$. This shows that

$$\mathrm{vert}(Q) = r + \mathrm{vert}(P)$$

and since every polytope is the convex hull of its vertices [5], we have

$$\begin{aligned} Q &= \mathrm{conv}(\mathrm{vert}(Q)) \\ &= \mathrm{conv}(r + \mathrm{vert}(P)) \\ &= r + \mathrm{conv}(\mathrm{vert}(P)) \\ &= r + P \end{aligned}$$

□

Acknowledgements

Research for this thesis was completed during the Summer 2005 Mount Holyoke College Research Experience for Undergraduates (REU) and was funded by NSF grant #DMS-0353700. I would like to acknowledge the members of my REU group, Vince Lyzinski, Lisa Byrne, Sarah Gilles and Frances Worek for their support. Thanks to Professor Cattani for putting me in touch with this research group. Professor Braden's lectures helped me understand how the topic of this thesis relates to the theory of Toric Varieties.A special thanks is reserved for Professor Sidman. The generous help she provided me was indispensable. Without her guidance, this thesis could not have been completed.

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