ALGORITHMS FOR EVEN-SIDED SELFAGONS

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ABSTRACT. Let $P \in \mathbb{R}^2$ be a lattice polytope. Then P is smooth if for each vertex $v \in P$, $\{w_i - v\}$ forms part of a $\mathbb{Z}$-basis for $\mathbb{Z}^2$, where w_i is the first lattice vector along an edge incident at v. We say P has lattice-free edges if the only lattice points on the edges of P are the vertices of P. We present two new algorithms for constructing smooth lattice polytopes with lattice-free edges and an even number of edges. The equivalence of one of the algorithms to a previous algorithm is also proved using the inner normal fans of the polytopes.

1. Introduction

As part of an REU at Mount Holyoke College, a group of participants studied smooth lattice polytopes with lattice-free edges, called selfatopes, and related them to toric varieties. This type of polytope first appeared in a paper by Jessica Sidman and David Cox [1].

The research that we will present in this paper stems from two broad questions about selfatopes in the plane, called selfagons. First, given a particular n, can we construct a selfagon with n edges? How small, in terms of the number of interior lattice points, can a selfagon with n edges be? In trying to find an algorithm giving smaller selfagons than ones already found, Algorithms 3.1 and 4.1 for larger selfagons were constructed.

The work presented here also developed from the question of equivalence of selfagons with the same number of sides. Attempts to find an algorithm giving selfagons that were not equivalent to ones we had previously found led to Algorithm 5.1 that produced equivalent selfagons to those from Algorithms 6.1 and 6.3. This equivalence is shown through the use of inner normal fans.

Preliminary definitions will be given in §2, followed by the presentation of Algorithms 3.1 and 4.1 in §3 and §4. In §5, Algorithm 5.1 will be explained. After a review of Algorithms 6.1 and 6.3 in §6, the equivalence of Algorithm 5.1 to Algorithms 6.1 and 6.3 is proved in §7.

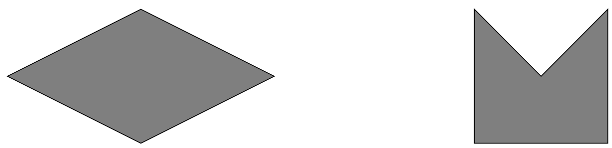
Research supported by NSF grant # DMS-0353700.

2. Preliminaries

First we will review definitions needed for the study of selfagons. For further study of polytopes, refer to [2], [7], [6], and [3].

Definition 2.1. A set A is *convex* if for every $p, q \in A$, $tp+(1-t)q \in A$ for $0 \leq t \leq 1$.

Example 2.2. The set of points on the left is convex, while the set of points on the right is not.



Definition 2.3. The *convex hull* of a set $A \in \mathbb{R}^n$ is the intersection of all convex sets containing A.

Definition 2.4. A *polytope* is the convex hull of a finite set of points in $\mathbb{R}^n$.

Example 2.5. The polytope on the right is the convex hull of the finite set of points on the left.

Definition 2.6. A *lattice polytope* is a polytope whose vertices all have integer coordinates.

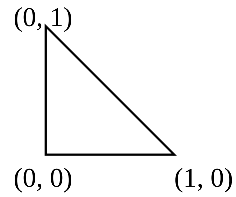
Definition 2.7. Let P be a lattice polytope. P has *lattice-free edges* if the only lattice points on the edges of P are the vertices of P.

Definition 2.8. Let $P \in \mathbb{R}^n$ be a lattice polytope. Then P is *smooth* if for each vertex $v \in P$, $\{w_i - v\}$ forms part of a $\mathbb{Z}$-basis for $\mathbb{Z}^n$ where w_i is the first lattice vector along an edge incident at v. If k, the number of edges incident to v, equals n, then $\{w_i - v\}$ forms a basis and $\begin{vmatrix} w_1 - v \\ \vdots \\ w_k - v \end{vmatrix} = \pm 1$.

Example 2.9. The simplex below is a smooth polytope since

$$\begin{vmatrix}(1,0)-(0,0)\\(0,1)-(0,0)\end{vmatrix}=\begin{vmatrix}1&0\\0&1\end{vmatrix}=1,\ \begin{vmatrix}(0,1)-(1,0)\\(0,0)-(1,0)\end{vmatrix}=\begin{vmatrix}-1&1\\-1&0\end{vmatrix}=1, \text{ and}$$

$$\begin{vmatrix}(0,0)-(0,1)\\(1,0)-(0,1)\end{vmatrix}=\begin{vmatrix}0&-1\\1&-1\end{vmatrix}=1.$$



Definition 2.10. A smooth, lattice-free lattice polytope is called a *selfatope*. In $\mathbb{R}^2$, this polytope is called a *selfagon*. Example 2.9 is a selfagon.

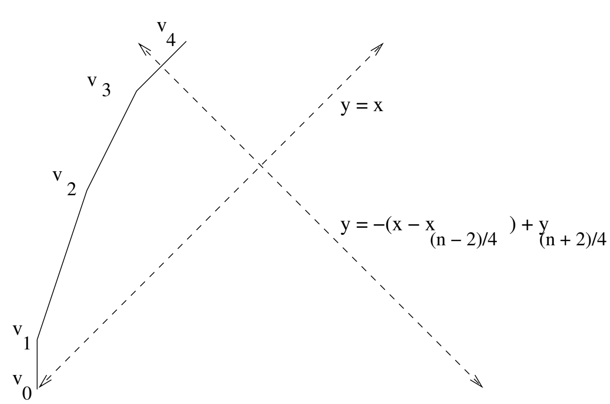
Throughout this paper, we will use the following convention:

Convention 2.11. Let P be a n-gon. Label its vertices by v_i clockwise with $v_n = v_0$ and $v_{n+1} = v_1$.

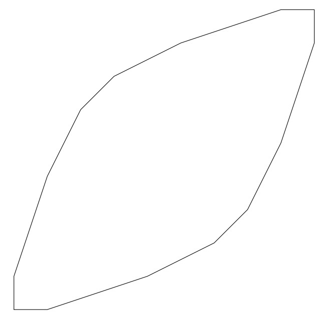
3. Method for constructing n-sided selfagons, $2 \mid n$, $4 \nmid n$

Algorithm 3.1. Begin with $v_0 = (0,0), v_1 = (0,1)$. For $i = 2, \ldots, \frac{n+2}{4}$, let $v_i = v_{i-1} + (1, \frac{n-2}{4} - i + 2)$. We finish construction of the polytope by reflections of this segment of the polytope: first reflect about the line $y = x$ and then about the line $y = -(x - x_{\frac{n-2}{4}}) + y_{\frac{n+2}{4}}$, where $v_i = (x_i, y_i)$.

Example 3.2. The 14-gon We start with $v_0 = (0,0)$ and $v_1 = (0,1)$. Since $\frac{n+2}{4} = 5$, we need to compute $v_2, v_3,$ and v_4. From our recursive formula we obtain $v_2 = v_1 + (1,3) = (1,4), v_3 = v_2 + (1,2) = (2,6), v_4 = v_3 + (1,1) = (3,7)$.



We then reflect this segment about $y = x$ and $y = -(x-2)+7$ to obtain the 14-gon pictured below.



Proposition 3.3. *The polygons constructed by Algorithm 3.1 are self-agons.*

Proof. By construction, these polygons are clearly lattice polygons with lattice free edges. We shall prove that these polygons are also smooth.

We will prove the smoothness of these polygons by showing that

$$\begin{vmatrix} v_{i+1} - v_i \\ v_{i-1} - v_i \end{vmatrix} = \pm 1$$

for each vertex v_i.

First,

$$\begin{vmatrix} v_1 - v_0 \\ v_{n-1} - v_0 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1.$$

Similarly,

$$\begin{vmatrix} v_2 - v_1 \\ v_0 - v_1 \end{vmatrix} = \begin{vmatrix} 1 & \frac{n-2}{4} \\ 0 & -1 \end{vmatrix} = -1$$

Next consider v_i with $i = 2, \ldots, \frac{n-2}{4}$. Note that $v_i = v_{i-1} + (1, \frac{n-2}{4} - i + 2)$ and $v_{i+1} = v_i + (1, \frac{n-2}{4} - (i+1) + 2)$, which gives us $v_{i+1} - v_i = (1, \frac{n-2}{4} - (i+1) + 2)$ and $v_{i-1} - v_i = -(v_i - v_{i-1}) = (-1, -(\frac{n-1}{2} - i + 2))$. Thus

$$\begin{vmatrix} 1 & \frac{n-2}{4} - (i-1) + 2 \\ -1 & -(\frac{n-2}{4} - i + 2) \end{vmatrix} = 1.$$

Now considering $v_{\frac{n+2}{4}}$, we see that $v_{\frac{n-2}{4}} - v_{\frac{n+2}{4}} = (-1, -1)$ and $v_{\frac{n+6}{4}} - v_{\frac{n+2}{4}} = (2, 1)$. Thus

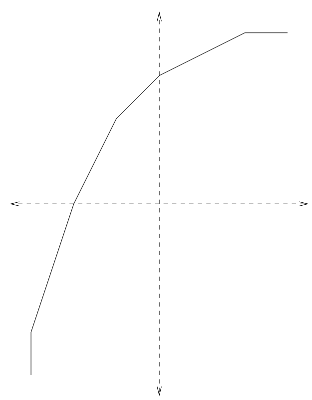
$$\begin{vmatrix} -1 & -1 \\ 2 & 1 \end{vmatrix} = 1.$$

All other v_i with adjacent v_{i+1}, v_{i-1} will have the same determinant as one of the cases above, up to a sign, since all other v_{i-1}, v_i, v_{i+1} segments of the polygon are reflections and/or rotations of the previously determined segments. Thus the polygon constructed is smooth and a selfagon. □

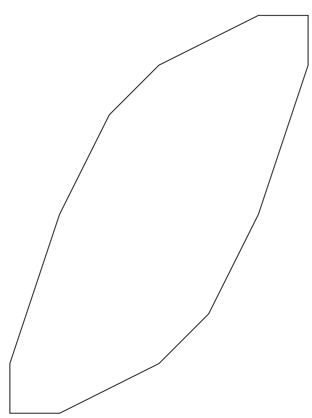
4. New method for constructing n-sided selfagons, $4 \mid n$, $n \geq 12$

Algorithm 4.1. Begin with $v_0 = (0,0), v_1 = (0,1)$. For $i = 2, \ldots, \frac{n+4}{4}$, let $v_i = v_{i-1} + (1, \frac{n}{4} + 2 - i)$. For $i = \frac{n+8}{4}, \ldots, \frac{n-2}{2}$, let $v_i = v_{i-1} + (i - \frac{n}{4}, 1)$. Let $v_{\frac{n}{2}} = v_{\frac{n-2}{2}} + (1,0)$. To finish construction of the polygon, reflect the segment already constructed about the lines $y = \frac{1}{2} y_{\frac{n}{2}}$ and $x = \frac{1}{2} x_{\frac{n}{2}}$.

Example 4.2. We will construct the 12-gon as an example. Let $v_0 = (0,0)$ and $v_1 = (0,1)$. Since $\frac{n+4}{4} = 4$, we first calculate v_i for $i = 2, \ldots, 4$. Using our formula, we find that $v_2 = v_1 + (1,3) = (1,4)$, $v_3 = v_2 + (1,2) = (2,6)$, and $v_4 = v_3 + (1,1) = (3,7)$. For v_5, we have $v_5 = v_4 + (2,1) = (5,8)$. Lastly, $v_6 = v_5 + (1,0) = (6,8)$.



We then reflect this segment about the lines $y = 4$ and $x = 3$ to complete the polytope.



Proposition 4.3. *The polgons constructed from Algorithm 4.1 are selfagons.*

Proof. As before, the polygon we have constructed is clearly a lattice polytope with lattice free edges, with a simple calculation to prove smoothness of the polytope.

First,

$$\begin{vmatrix} v_1 - v_0 \\ v_{n-1} - v_0 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 1.$$

For v_1, we have

$$\begin{vmatrix} v_2 - v_1 \\ v_0 - v_1 \end{vmatrix} = \begin{vmatrix} 1 & \frac{n}{4} \\ 0 & -1 \end{vmatrix} = -1.$$

For v_i, with $i = 2, \dots, \frac{n}{4}$, note that $v_{i+1} - v_i = (1, \frac{n}{4} + 1 - i)$ and $v_{i-1} - v_i = -(v_i - v_{i-1}) = (1, \frac{n}{4} + 2 - i)$. Thus

$$\begin{vmatrix} v_{i+1} - v_i \\ v_{i-1} - v_i \end{vmatrix} = \begin{vmatrix} 1 & \frac{n}{4} + 1 - i \\ 1 & \frac{n}{4} + 2 - i \end{vmatrix} = 1.$$

Next, consider $v_{\frac{n+4}{4}}$. Note that $v_{\frac{n}{4}} - v_{\frac{n+4}{4}} = (-1, -1)$ and $v_{\frac{n+8}{4}} - v_{\frac{n+4}{4}} = (2, 1)$. Thus

$$\begin{vmatrix} -1 & -1 \\ 2 & 1 \end{vmatrix} = 1.$$

The determinants for all other v_{i-1}, v_i, v_{i+1} segments are the same as one of the previously calculated vertices, up to a sign, since all other v_{i-1}, v_i, v_{i+1} segments of the polygon are a relfection and/or rotation of the cases already considered. Thus the polygon is smooth and a selfagon. □

5. Method for Constructing Even n-sided selfagons, with $n \geq 10$

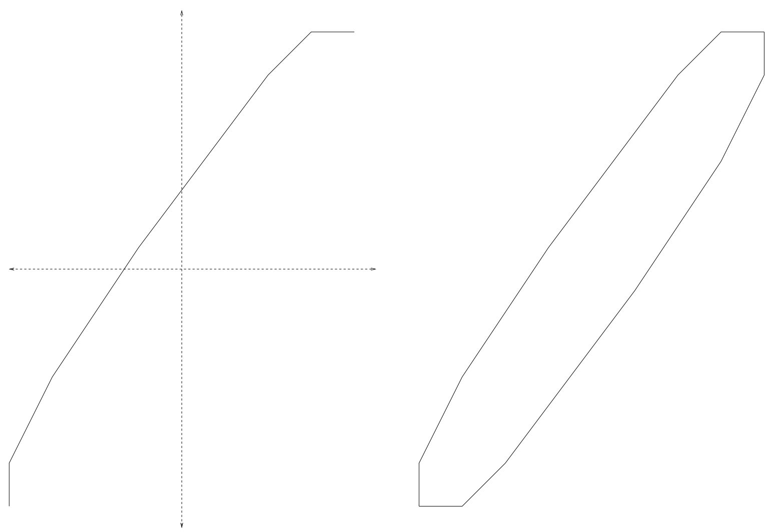
Algorithm 5.1. Begin with $v_0 = (0, 0)$. For $i = 1, \dots, \frac{n-4}{2}$, let $v_i = v_{i-1} + (i - 1, i)$. Define $v_{\frac{n-4}{2}+1} = v_{\frac{n-4}{2}} + (1, 1)$ and $v_{\frac{n-4}{2}+2} = v_{\frac{n-4}{2}+1} + (1, 0)$. Reflect this segment about the line $y = \frac{1}{2} y_{\frac{n}{2}}$ and $x = \frac{1}{2} x_{\frac{n}{2}}$to complete the polygon.

Example 5.2. Constructing the 12-gon

Using the above algorithm, we compute the first 6 vertices of a 12-gon:

i	v_i
0	(0, 0)
1	(0, 1)
2	(1, 3)
3	(3, 6)
4	(6, 10)
5	(7, 11)
6	(8, 11)

The segment of the 12-gon constructed by the vertices is pictured below on the left. Rotating the segment about the line $y = \frac{11}{2}x$ and $x = 4$ gives us a 12-gon, pictured on the right.



Proposition 5.3. *The polygons constructed by Algorithm 5.1 are selfagons.*

Proof. By construction, the polygon is clearly lattice with lattice free edges. A series of simple calculations shows that the polygon is also smooth. We will show that for each vertex v_i,

$$\begin{vmatrix} v_{i+1} - v_i \\ v_{i-1} - v_i \end{vmatrix} = \pm 1.$$

For v_0,

$$\begin{vmatrix} v_1 - v_0 \\ v_{n-1} - v_0 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1.$$

For v_1,

$$\begin{vmatrix} v_2 - v_1 \\ v_0 - v_1 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} = -1.$$

Note that for $i = 2, \ldots, \frac{n-4}{2} - 1$, $v_{i+1} = v_i + (i, i+1)$ and $v_i = v_{i-1} + (i-1, i)$. Thus

$$\begin{vmatrix} v_{i+1} - v_i \\ v_{i-1} - v_i \end{vmatrix} = \begin{vmatrix} i & i+1 \\ -(i-1) & -i \end{vmatrix} = -1.$$

For $v_{\frac{n-4}{2}}$,

$$\begin{vmatrix} v_{\frac{n-4}{2}+1} - v_{\frac{n-4}{2}} \\ v_{\frac{n-4}{2}-1} - v_{\frac{n-4}{2}} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -(\frac{n-4}{2} - 2) & -(\frac{n-4}{2} - 1) \end{vmatrix} = -1.$$

For $v_{\frac{n-4}{2}+1}$,

$$\begin{vmatrix} v_{\frac{n-4}{2}+2} - v_{\frac{n-4}{2}+1} \\ v_{\frac{n-4}{2}} - v_{\frac{n-4}{2}+1} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -1 & -1 \end{vmatrix} = -1.$$

Because of the symmetry of the polygon, it is sufficient to only check these determinants for $i = 0, \ldots, \frac{n-4}{2} + 1$. Thus we conclude that the polygon is smooth and also a selfagon. □

6. Review of the Lyzinski Algorithm

The Lyzinski algorithm also constructs even selfagons using two cases: even polygons divisible by 4, and even selfagons not divisible by 4. The Lyzinski algorithm is presented here in a new manner which will aid in later calculations.

Algorithm 6.1. The Lyzinski Algorithm for the n-gon, $4 \mid n$

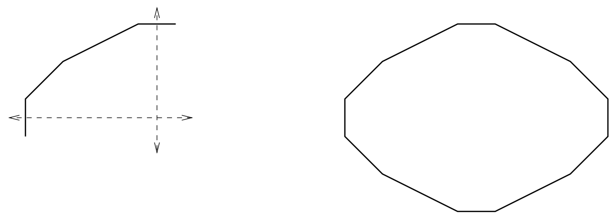
Set $v_0 = (0, 0)$. Let $v_i = v_{i-1} + (i-1, 1)$ for $1 \le i \le \frac{n}{4}$. Define $v_{\frac{n+4}{4}} = v_{\frac{n}{4}} + (1, 0)$. To finish construction of the polygon, reflect this segment about the line $y = \frac{1}{2}$ and $x = x_{\frac{n+4}{4}} - \frac{1}{2}$. To aid in later computations, we will need explicit formulas for vertices v_i, $\frac{n+8}{4} \le i \le \frac{n}{2}$. This formula is $v_i = v_{i-1} + (\frac{n}{2} + 1 - i, -1)$ for $\frac{n+8}{4} \le i \le \frac{n}{2}$.

Example 6.2. The Lyzinski 12-gon

We calculate the first 5 vertices of the Lyzinski 12-gon.

i	v_i
0	(0, 0)
1	(0, 1)
2	(1, 2)
3	(3, 3)
4	(4, 3)

The segment of the selfagon formed from these vertices is pictured below on the left along with the axes of rotation to form the Lyzinski 12-gon, pictured on the right.



Algorithm 6.3. The Lyzinski Algorithm for the n-gon, $2 \mid n$, $4 \nmid n$

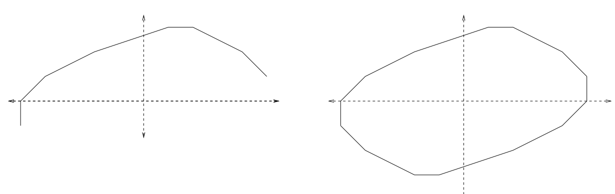
Set $v_0 = (0,0)$. Let $v_i = v_{i-1} + (i-1, 1)$ for $1 \leq i \leq \frac{n+2}{4}$. Let $v_{\frac{n+6}{4}} = v_{\frac{n+2}{4}} + (1,0)$. For $\frac{n+10}{4} \leq j \leq \frac{n}{2}$, let $v_j = v_{j-1} + (\frac{n}{2} + 1 - j, -1)$. Reflect this segment about the lines $y = 1$ and $x = \frac{1}{2}x_{\frac{n}{2}}$ to complete the selfagon, pictured on the right.

Example 6.4. The Lyzinski 14-gon

We compute the first 8 vertices of the Lyzinski 14-gon:

i	v_i
0	(0, 0)
1	(0, 1)
2	(1, 2)
3	(3, 3)
4	(6, 4)
5	(7, 4)
6	(9, 3)
7	(10, 2)

The segment of the selfagon constructed with these points is pictured on the left in the figure below. Reflect this segment about the lines $y = 1$ and $x = 5$ to complete the 14-gon.



7. Showing that Algorithm 5.1 and the Lyzinski algorithm construct equivalent selfagons

Our goal is to find a linear transformation between selfagons of the same number of sides constructed with the Lyzinski algorithm (Algorithms 6.1 and 6.3) and Algorithm 5.1. We begin by defining notation for transformations of polytopes and equivalent selfatopes.

Definition 7.1. Let P be a polytope and A be a linear transformation. We define $AP = \{Ap : p \in P\}$.

Definition 7.2. Let $P, Q \in \mathbb{R}^n$ be selfatopes. We say that P and Q are *equivalent selfatopes* if there exists $A \in GL_n(\mathbb{Z})$ and a translation $r \in \mathbb{Z}^n$ such that $AP + r = Q$.

Definition 7.3. A *polyhedral cone* is defined as

$$\text{cone}(u_1, \ldots, u_k) = \{t_1u_1 + \cdots + t_ku_k : t_i \geq 0\}.$$

Definition 7.4. A *polyhedral fan* Δ is a finite union of cones satisfying

- If $\sigma \in \Delta$ and τ is a face of σ, then $\tau \in \Delta$.
- If $\sigma, \tau \in Delta$, then $\sigma \cap \tau$ is a face of each.

Definition 7.5. An *affine hyperplane* in $\mathbb{R}^n$ is the set of all solutions to a linear equation $a_1w_1 + \ldots + a_nw_n = \alpha$ where not all $a_i = 0$.

Definition 7.6. Let A be a subset of $\mathbb{R}^n$. The *affine hull* of A, $\text{aff}(A)$, is the intersection of all affine hyperplanes containing A.

Definition 7.7. The *dimension* of a polytope P is the dimension of its affine hull. A polytope is *full dimensional* in $\mathbb{R}^n$ if the dimension of P is n.

Definition 7.8. Let $P \in \mathbb{R}^n$ be a full-dimensional polytope. The *inner normal fan* Δ_P is the union over all faces $F \subset P$ of cones

$$\sigma_F = \{v \in \mathbb{R}^n : u \cdot v \leq u' \cdot v, \forall u \in F, \forall u' \in P\}.$$

Lemma 7.9. *Let P be a full-dimensional selfagon in $\mathbb{R}^2$ with inner normal fan Δ_P. If $B \in GL_2(\mathbb{Z})$ then the selfagon $(B^{-1})^TP$ has inner normal fan $B\Delta_P$.*

Proof. Note that $(B^{-1})^TP$ is a selfagon since $(B^{-1})^T \in GL_2(\mathbb{Z})$.

We are left to show that the inner normal fan of $(B^{-1})^TP$ is $B\Delta_P$. For $r_j, v_i \in \mathbb{R}^2$,

$$\begin{aligned}\langle r_j, v_i\rangle &= \langle B^{-1}Br_j, v_i\rangle \\ &= \langle Br_j, (B^{-1})^Tv_i\rangle.\end{aligned}$$

Let v_j be a vertex of P, with $v_{j+1} - v_j$ an edge of P. If we let r_j be the inner normal to $v_{j+1} - v_j$, the linear function $f(x) = \langle Br_j, x\rangle$ equals 0 at $x = (B^{-1})^T(v_{j+1} - v_j)$. Therefore Br_j is normal to $(B^{-1})^TP$. Since r_j is an inner normal to $v_{j+1} - v_j$, for $k \neq j$,

$$\begin{aligned}&\langle r_j, v_{j+1} - v_j\rangle < \langle r_j, v_{k+1} - v_k\rangle \\ \iff &\langle Br_j, (B^{-1})^T(v_{j+1} - v_j)\rangle < \langle Br_j, (B^{-1})^T(v_{k+1} - v_k)\rangle.\end{aligned}$$

Thus Br_j is inner normal to $(B^{-1})^TP$. □

Corollary 7.10. *Let P, Q be full-dimensional selfagons in $\mathbb{R}^2$ with inner normal fans Δ_P and Δ_Q, respectively. If $B \in GL_2(\mathbb{Z})$ such that $B\Delta_P = \Delta_Q$, then $(B^{-1})^T P$ is equal to Q up to translation.*

Proof. From Theorem 7.9, $(B^{-1})^T P$ has inner normal fan $B\Delta_P$. From a proof by Aaron Wolbach, there is at most once selfagon up to translation with a given inner normal fan. Thus we conclude that $(B^{-1})^T P$ is equal to Q. □

Lemma 7.11. (Fulton [3]) *Let P be a polytope with inner normal fan Δ_Q. For each generating ray $r_i \in \Delta_P$ there exists a constant a_i such that $a_i r_i = r_{i-1} + r_{i+1}$.*

Lemma 7.12. *Let a_i be the constant such that $a_i r_i = r_{i-1} + r_{i+1}$ for rays r_{i-1}, r_i, and r_{i+1} of Δ_P. The constant a_i is invariant under a linear transformation of the rays of the inner normal fan.*

Proof. Let B be a linear transformation.

$$B(a_i r_i) = B(r_{i-1} + r_{i+1}) \iff a_i B(r_i) = Br_{i-1} + Br_{i+1}.$$

□

Proposition 7.13. *Algorithm 5.1 and the Lyzinski algorithm produce equivalent selfagons.*

Proof. Let P be a selfagon generated by the Lyzinski algorithm and Q be a selfagon generated by Algorithm 5.1. Using Corollary 7.10, we wish to find a linear transformation $B \in GL_2(\mathbb{Z})$ between the inner normal fans of the selfagons produced by the Lyzinski algorithm and Algorithm 5.1 to show that $(B^{-1})^T P$ is Q up to translation. Then we can conclude by the definition of equivalence of polytopes that P is equivalent to Q.

To find $B \in GL_2\mathbb{Z}$ such that $B\Delta_P = \Delta_Q$, we calculate the the generating rays s_i of Δ_Q and r_i of Δ_P and also the corresponding coefficients from Lemma 7.11. Since Lemma 7.12 tells us that these coefficients are invariants, we will use these coefficients in determining the ray r_j that s_i maps to under B.

Claim 1: Let $s_i \in \Delta_Q$ be the inner normal to the edge $w_{i+1} - w_i$ of Q. For $1 \le i < \frac{n-4}{2}$, $s_i = (i+1, -i)$.

Proof. Since $s_i \cdot (w_{i+1} - w_i) = 0$ and $w_{i+1} = w_i + (i, i+1)$, then either $s_i = (i+1, -i)$ or $s_i = (-i-1, i)$. Let $w_i = (x_i, y_i)$. For $1 \le i < \frac{n-4}{2}$, so $x_i < x_{i+1}$ and $y_i < y_{i+1}$. Therefore, for s_i to be *inner* normal to the edge $w_{i+1} - w_i$, s_i must have a positive x-coordinate and negative y-coordinate.

We will need s_i with $0 \leq i \leq \frac{n}{2}$ and $i = n-1$ to calculate the b_i for $0 \leq i \leq \frac{n-2}{2}$. We have not yet calculated s_i with $i = 0, \frac{n-4}{2}, \frac{n-2}{2}, \frac{n}{2}$ or $n-1$. Through further, more direct computations, we find that $s_0 = (1,0), s_{\frac{n-4}{2}} = (1,-1), s_{\frac{n-2}{2}} = (0,-1), s_{\frac{n}{2}} = (-1,0)$, and $s_{n-1} = (0,1)$.

Now we wish to find b_i for $0 \leq i \leq \frac{n-2}{2}$ such that $b_i s_i = s_{i-1} + s_{i+1}$ (Lemma 7.11). We calculate that $b_0 = 2, b_1 = 2, b_i = 2$ for $2 \leq i \leq \frac{n-8}{2}, b_{\frac{n-6}{2}} = 1, b_{\frac{n-4}{2}} = \frac{n-4}{2}$, and $b_{\frac{n-2}{2}} = 1$.

Claim 2: Let P be an n-gon generated by the Lyzinski Algorithm with $4 \mid n$. Let $r_i \in \Delta_P$ be the inner normal to the edge $v_{i+1} - v_i$ of P. For $1 \leq i < \frac{n-4}{4}$, $r_i = (1, -i)$, and for $\frac{n+8}{4} \leq i \leq \frac{n-2}{2}$, $r_i = (-1, -\frac{n}{2} + i)$.

Proof. Since $r_i \cdot (v_{i+1} - v_i) = r_i \cdot (i, 1) = 0$, r_i must equal $(1, -i)$ or $r_i = (-1, i)$. For $1 \leq i \leq \frac{n-4}{4}$, note that $x_i < x_{i+1}$ and $y_i < y_{i+1}$ for $v_i = (x_i, y_i), v_{i+1} = (x_{i+1}, y_{i+1})$ since $v_i = v_{i-1} + (i-1, 1)$. Therefore, for r_i to be *inner* normal to the edge between v_i and v_{i+1}, we see that r_i must have a positive x coordinate and a negative y coordinate. Hence we take $r_i = (1, -i)$ when $1 \leq i < \frac{n}{4}$.
We then compute r_i for $\frac{n+8}{4} \leq i \leq \frac{n-2}{2}$. We want $r_i \cdot (v_{i+1} - v_i) = r_i \cdot (\frac{n}{2} - i, -1) = 0$, which implies that $r_i = (1, \frac{n}{2} - i)$ or $r_i = (-1, -\frac{n}{2} + i)$. The fact that $x_i < x_{i+1}$ and $y_i < y_{i+1}$ requires us to choose $r_i = (-1, -\frac{n}{2} + i)$.

Through the same method, we find $r_0 = (1,0), r_{\frac{n}{4}} = (0,-1)$, and $r_{\frac{n+4}{4}} = (-1, -\frac{n-4}{4}), r_{\frac{n}{2}} = (-1, 0)$ and $r_{\frac{n+2}{2}} = (-1, 1)$.

Continuing the case for the n-gon with $4 \mid n$, we compute the a_i such that $a_i r_i = r_{i-1} + r_{i+1}$. We find that $a_0 = 1, a_1 = 2, a_i = 2$ for $2 \leq i \leq \frac{n-8}{4}, a_{\frac{n-4}{4}} = 1, a_{\frac{n}{4}} = \frac{n-4}{2}, a_{\frac{n+4}{4}} = 1, a_{\frac{n+8}{4}} = 2, a_i = 2$ for $\frac{n+12}{4} \leq i \leq \frac{n-4}{2}, a_{\frac{n-2}{2}} = 2$, and $a_{\frac{n}{2}} = 2$.

We see that $a_{\frac{n}{4}}$ and $b_{\frac{n-4}{2}}$ are the only two coefficients that equal $\frac{n-4}{2}$. By Lemma 7.12, these coefficients are invariants. Hence we look for a linear transformation from Δ_P to Δ_Q that maps $r_{\frac{n}{4}}$ to $s_{\frac{n-4}{2}}$ and in general, maps r_i to $s_{i + \frac{n-8}{2}}$. Solving for a linear transformation B such that $Br_0 = s_{\frac{n-8}{4}}$ and $Br_1 = s_{\frac{n-4}{4}}$ gives us $B = \begin{pmatrix} \frac{n-4}{4} & -1 \\ -\frac{n-8}{4} & 1 \end{pmatrix}$. By Corollary 7.10, $(B^{-1})^T = \begin{pmatrix} 1 & \frac{n-8}{4} \\ 1 & \frac{n-4}{4} \end{pmatrix}$ is a matrix such that $(B^{-1})^T P$ is Q up to translation. Thus we have shown that for P is equivalent to Q, when P is an n-gon with $4 \mid n$.

Claim 3: Let P be an n-gon generated by the Lyzinski Algorithm with $4 \nmid n$. Let $r_i \in \Delta_P$ be the inner normal to the edge $v_{i+1} - v_i$ of P. For $1 \leq i < \frac{n+2}{4}$, $r_i = (1, -i)$, and for $\frac{n+6}{4} \leq i < \frac{n}{2}$, $r_i = (-1, -\frac{n}{2} + i)$.

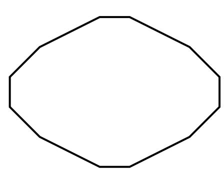
Proof. For $1 \leq i < \frac{n+2}{4}$, the same calculation as used for the n-gon with $4 \mid n$ gives us $r_i = (1, -i)$. For $\frac{n+6}{4} \leq i < \frac{n}{2}$, we want $r_i \cdot (v_{i+1} - v_i) = r_i \cdot (\frac{n}{2} - i, -1) = 0$. Thus $r_i = (1, \frac{n}{2} - i)$ or $r_i = (-1, -\frac{n}{2} + i)$. Using a similar argument as before, since $x_i < x_{i+1}$ and $y_i > y_{i+1}$, r_i must have both a negative x and y coordinate. Hence r_i must equal $(-1, -\frac{n}{2} + i)$.

Through similar computations we also find that $r_0 = (1, 0), r_{\frac{n+2}{4}} = (0, -1), r_{\frac{n}{2}} = (-1, 0)$, and $r_{n-1} = (1, 1)$.

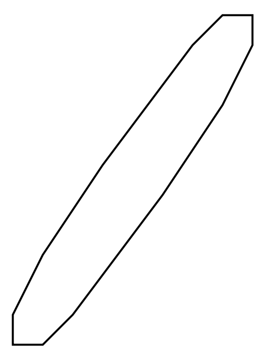
Calculating the $a_i's$ for the n-gon with $4 \nmid n$ gives us the following: $a_0 = 2, a_1 = 2, a_i = 2$ for $2 \leq i \leq \frac{n-6}{4}, a_{\frac{n-2}{4}} = 1, a_{\frac{n+2}{4}} = \frac{n-4}{2}, a_{\frac{n+6}{4}} = 1, a_j = 2$ for $\frac{n+10}{4} \leq j \leq \frac{n-4}{2}$, and $a_{\frac{n-2}{2}} = 2$.

We wish to find a transformation matrix for the transformation between Δ_P and Δ_Q. From the computed inner normals for both fans, since the invariants $a_{\frac{n+2}{4}}$ and $b_{\frac{n-4}{2}}$ are the only constants that equal $\frac{n-4}{2}$, we look for a transformation which maps $r_{\frac{n+2}{4}}$ to $s_{\frac{n-4}{2}}$, and in general, maps r_i to $s_{i+\frac{n-6}{4}}$. To form the transformation matrix B, solve for B such that $Br_0 = s_{\frac{n-6}{4}}$ and $Br_1 = s_{\frac{n-2}{4}}$. We find that $B = \begin{pmatrix} \frac{n-10}{4} & -1 \\ -\frac{n-6}{4} & 1 \end{pmatrix}$ and $(B^{-1})^T = \begin{pmatrix} 1 & \frac{n-10}{4} \\ 1 & \frac{n-6}{4} \end{pmatrix}$. Thus P is equivalent to Q, when P is an n-gon with $4 \nmid n$ □

Example 7.14. We shall show that the 12-gons generated by these two algorithms are equivalent. The 12-gon on the left, P is generated by the Lyzinski algorithm (Algorithm 6.1), and the 12-gon on the right, Q, is generated by 5.1.



P

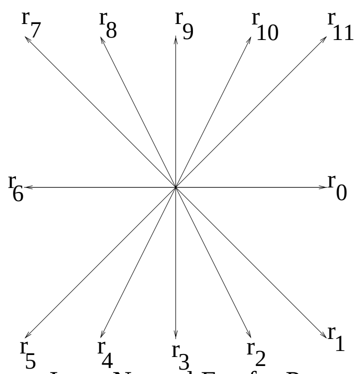


Q

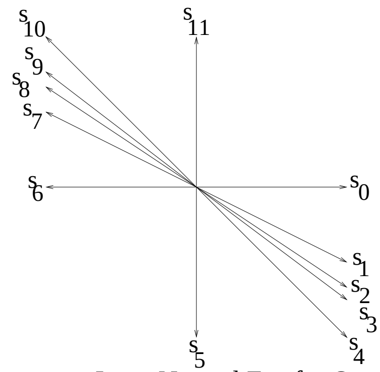
The generating rays and corresponding constant coefficients from Lemma 7.11 of Δ_P and Δ_Q are listed below, followed by Δ_P and Δ_Q

i	r_i	a_i
0	(1, 0)	2
1	(1, -1)	2
2	(1, -2)	1
3	(0, -1)	4
4	(-1, -2)	1
5	(-1, -1)	2
6	(-1, 0)	2
7	(-1, 1)	2
8	(-1, 2)	1
9	(0, 1)	4
10	(1, 2)	1
11	(1, 1)	2

i	s_i	b_i
0	(1, 0)	2
1	(2, -1)	2
2	(3, -2)	2
3	(4, -3)	1
4	(1, -1)	4
5	(0, -1)	1
6	(-1, 0)	2
7	(-2, 1)	2
8	(-3, 2)	2
9	(-4, 3)	1
10	(-1, 1)	4
11	(0, 1)	1



Inner Normal Fan for P



Inner Normal Fan for Q

Let B be the matrix of transformation from Δ_P to Δ_Q. Since $a_3 = 4$ and $b_4 = 4$, we will calculate a transformation which maps r_i to s_{i+1}. We calculate B by choosing a r_i and r_j and solving for B such that $Br_i = s_{i+1}$ and $Br_j = s_{j+1}$. We find that $B = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ and $(B^{-1})^T = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$.

We find that $(B^{-1})^T P$ applied to the selfagon generated by the Lyzinski algorithm gives us a translation of Q, the selfagon generated by Algorithm 5.1.

8. Acknowledgements

I would like to thank my research advisor Dr. Jessica Sidman for her guidance throughout the research process. My gratitude goes to Vincent Lyzinski, author of the Lyzinski algorithm, and to Aaron Wolbach for his proof which was used in this paper. I would also like to acknowledge the help of Sarah Gilles in the compilation of some of the definitions for the Preliminaries section. Thank you to the final member of the Polytopes and Toric Varieties research group, Lisa Byrne, the p-Adic Numbers research group, and Margaret Robinson for making MHC REU 2005 a wonderful experience. Thank you also to the NSF for funding MHC REU. I also am grateful to the makers of *Polymake* [4] and *Macaulay 2*[5], whose programs were used in the study of polytopes and toric varieties.

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