THE NATURAL BOUNDARY OF THE EULER PRODUCT OF LOCAL ZETA FUNCTION ASSOCIATED WITH THE POLYNOMIAL $f(x,y)=x^2-y^3$

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Abstract. We find the Igusa local zeta function for the polynomial $f(x,y)=x^2-y^3$ using the stationary phase formula. Next we determine the natural boundary for the Euler product of this zeta function. This work was completed under the supervision of Professor Margaret Robinson as part of the Mount Holyoke Summer Mathematics Institute, an NSF funded REU Program.

1. Introduction

The main result of this paper is the computation of the natural boundary of the Euler product of the Igusa local zeta function for the polynomial $f(x,y)=x^2-y^3$, using the method introduced by Marcus du Sautoy in [**?**] . Our research is motivated by the bigger problem of exploring a possible relation between the singularities of Igusa local zeta functions and their natural boundaries. This is an important polynomial to look at since its the first non-trivial example used to study the resolution of singularities. We have also included the calculation of the Igusa local zeta function for the polynomial $f(x,y)=x^2-y^3$, using the stationary phase formula.

2. Preliminaries

Our local zeta functions are integrals over the ring of p-adic integers. To understand the p-adic numbers we need to look at the p-adic absolute value which is defined as:

$$|x|_p = \begin{cases} p^{-\mathrm{ord}_p x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

where p is prime and $\mathrm{ord}_p x$ is defined as the smallest power of p with a non-zero coefficient in the p-adic expansion of x. When we complete the rational numbers with respect to the p-adic absolute value we get the field of p-adic numbers, denoted by $\mathbb{Q}_p$. In this field every element is represented uniquely by a p-adic expansion of the form:

$$x = a_{-n}p^{-n} + a_{-n+1}p^{-n+1} + ... + a_0 + a_1p + a_2p^2 + ...$$

for all $a \in \{\ 0,1,....p-1\}$. The p-adic integers are the subring of $\mathbb{Q}_p$ given by:

$$\mathbb{Z}_p = \{\ x \in \mathbb{Q}_p : |x|_p \leq 1\}$$

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Hence each p-adic integer has a p-adic expansion of the form $x = a_0 + a_1 p + a_2 p^2 + ...$, for $a_i \in \{\ 0, 1, ..., p-1\}$ [?].

In the first section of this paper we compute the Igusa local zeta function of the polynomial $f(x, y) = x^2 - y^3$. For any polynomial $f(x) = f(x_1, x_2, ..., x_n)$ for n variables with coefficients in $\mathbb{Z}_p$, the Igusa local zeta function is defined as:

$$Z(s) = \int_{\mathbb{Z}_p^n} |f(x)|_p^s dx$$

for $Re(s) > 0$. Note that quite often, we think of $Z(s)$ as a function of $t = p^{-s}$ since for a given value of p^{-s}, s is not uniquely defined. Via an associated Poincare series, the Igusa local zeta function counts the number of solutions modulo p^e of the congruences $f(x) \equiv \bmod p^e$, for $e = 1, 2, 3, ...$ [?].

In many cases, the Igusa local zeta function of a polynomial $f(x)$ in n variables with coefficients in $\mathbb{Z}_p$ can be calculated by the stationary phase formula (SPF) [?]. The formula allows us to compute the Igusa local zeta function if we understand the singularities modulo p of $f(x)$. The stationary phase formula states that:

$$Z(t) = (p^n - |\bar{N}|)p^{-n} + (|\bar{N}| - |\bar{S}|)p^{-n} t \left(\frac{1 - p^{-1}}{1 - p^{-1} t} \right) + \int_S |f(x)|_p^s dx$$

where,

$$\begin{aligned} \bar{f}(x) &\equiv f(x) \bmod p \\ n &= \text{number of variables in } \bar{f}(x) \\ \bar{N} &= \text{the vectors } x \text{ in } \mathbb{F}_p^n \text{ for which } \bar{f}(x) \equiv 0 \bmod \ p \\ |\bar{N}| &= \text{cardinality of } \bar{N} \\ \bar{S} &= \text{singular vectors } x \text{ in } \bar{N} \text{ such that all the partial derivatives of } \bar{f} \\ & \text{at } x \text{ are congruent to } 0 \bmod p \\ S &= \text{all vectors in } \mathbb{Z}_p^n \text{ which are congruent to vectors in } \bar{S} \text{ mod p} \\ t &= p^{-s} \end{aligned}$$

3. Computing $Z(t)$ for $f(x, y) = x^2 - y^3$

In this section we show the computation of the zeta function for the polynomial $f(x, y) = x^2 - y^3$. We have to apply the stationary phase formula four times to compute this particular zeta function.

Application 1
First, we must find the polynomial $\bar{f}$. Since there are no factors of p in the original polynomial, $\bar{f} = x^2 - y^3$.

What is $|\bar{N}|$?

$|\bar{N}|$ is p by the following argument.

Lemma. *There are p solutions to $x^2 - y^3 \equiv 0$ mod p in $\mathbb{F}_p$.*

Proof. There are $\frac{p-1}{2}$ squares in $\mathbb{F}_p$ because if $a^2 \equiv b$ mod p then $(-a)^2 \equiv b$ mod p is also true. Let r be a primitive root. Consider the set $A = \{r^{3k} | k \in \mathbb{Z}\}$.
Case 1: 3 and $p-1$ are relatively prime
If $r^{3k} \equiv 1$ mod p for some $k < p-1$, then 3 divides $p-1$ but we assume that 3 and $p-1$ are relatively prime, hence the smallest value for k is $p-1$. Therefore r^3 is a primitive root, so it must generate all residues mod p. Each residue is therefore a cube. Hence for each x_0 there exists a unique y_0 such that (x_0, y_0) is a solution. There are p choices for x_0, so we have p solutions in $\mathbb{F}_p$.
Case 2: 3 divides $p-1$
Then there are only $\frac{p-1}{3}$ elements in A and each cube in $\mathbb{F}_p$ must be an element of A. For each x there exists three elements y_1, y_2 and y_3 such that (x, y_i) is a solution for all i if and only if x is a cube. This is because if x is a cube, then there are three solutions y_1, y_2, and y_3 to $y^3 \equiv x^2$. Conversely, if y_1, y_2, and y_3 exist, then x^2 is a cube and hence x is a cube.

There are $\frac{p-1}{3}$ cubes in $\mathbb{F}_p$ and for each choice of x we have 3 choices for y. Hence there are $(\frac{p-1}{3})3 = p-1$ nonzero solutions, and therefore we have p solutions in $\mathbb{F}_p$. □

What is $|\bar{S}|$?
The partial derivatives of $\bar{f}$ with respect to each of the variables are:

$$\frac{\partial \bar{f}}{\partial x} = 2x \quad \frac{\partial \bar{f}}{\partial y} = -3y^2$$

Therefore, the only singular point occurs when $x = y = 0$, mod p for all p. Hence, $|\bar{S}|$=1. So now we use SPF to write out the terms of the zeta function for this polynomial:

$$\begin{aligned} Z(t) &= (p^n - |\bar{N}|)p^{-n} + (|\bar{N} - \bar{S}|)p^{-n} t \left(\frac{1-p^{-1}}{1-p^{-1}t}\right) + I_1 \\ Z(t) &= (p^2 - p)p^{-2} + (p-1)p^{-2} t \left(\frac{1-p^{-1}}{1-p^{-1}t}\right) + I_1 \\ Z(t) &= (1-p^{-1}) + (p^{-1} - p^{-2}) t \left(\frac{1-p^{-1}}{1-p^{-1}t}\right) + I_1 \end{aligned}$$

where I_1 is the integral over the singular point which we have to compute separately. Since the integral is evaluated over the singular points, x and y must all be congruent to 0 mod p. So x and y must have p-adic expansion of the form: $0p^0 + a_1 p^1 + a_2 p^2 \ldots$. We can write I_1 as:

$$\begin{aligned} I_1 &= \int_{(0,0)+p\mathbb{Z}_p^2} |(x^2 - y^3)|_p^s dx dy \\ &= \int_{p\mathbb{Z}_p^2} |(x^2 - y^3)|_p^s dx dy. \end{aligned}$$

Next, making the change of variables:

$$x = pu \qquad y = pv$$

we get the change in measure:

$$dx = p^{-1}du \qquad dy = p^{-1}dv$$

Our integral becomes:

$$\begin{aligned} I_1 &= p^{-2}\int_{\mathbb{Z}_p^2} |(p^2u^2 - p^3v^3)|_p^s dudv \\ &= p^{-2-2s}\int_{\mathbb{Z}_p^2} |(u^2 - pv^3)|_p^s dudv \\ &= p^{-2}t^2 Z_2(t). \end{aligned}$$

Now that I_1 is in the form of an integral evaluated over $\mathbb{Z}_p^2$, namely $Z_2(t)$, we can apply SPF to the new new polynomial $f_1(u,v) = u^2 - pv^3$.

Application 2

Now we must find the zeta function which we called $Z_2(t)$, signifying that we are starting our second SPF application. Notice that the $p^{-2}t^2$ that appeared in front of the integral is not included in $Z_2(t)$. For the purpose of computing the first two terms of the SPF, we need only to consider $\bar{f} = u^2$.

What is $|\bar{N}|$?
$|\bar{N}| = 1$ because there is only one value for which $\bar{f} \equiv 0 \bmod p$, namely when $u = 0$.

What is $|\bar{S}|$?
$|\bar{S}| = 1$ because$\frac{\partial \bar{f}}{\partial u} = 2u$ which equals 0 mod p only when $u = 0$, again assuming that $p \neq 2, 3$. Now we can use this information in SPF, where we let I_3 equal the polynomial integrated over the singular points:

$$\begin{aligned} Z_2(t) &= (p^n - |\bar{N}|)p^{-n} + (|\bar{N} - \bar{S}|)p^{-n}t\left(\frac{1-p^{-1}}{1-p^{-1}t}\right) + I_2 \\ &= (p-1)p^{-1} + (1-1)p^{-n}t\left(\frac{1-p^{-1}}{1-p^{-1}t}\right) + I_2 \\ &= (1-p^{-1}) + I_2 \end{aligned}$$

We now evaluate the I_2 term:

$$I_2 = \int_{\mathbb{Z}_p}\int_{0+p\mathbb{Z}_p} |u^2 - pv^3|_p^s dudv$$

We can make the change in variables: $u = pz$ and $du = p^{-1}dz$ and continue simplifying:

$$\begin{aligned} I_2 &= p^{-1}\int_{\mathbb{Z}_p^2} |p^2z^2 - pv^3|_p^s dzdv \\ &= p^{-1}t\int_{\mathbb{Z}_p^2} |pz^2 - v^3|_p^s dzdv \end{aligned}$$

$$= \quad p^{-1}tZ_3(t)$$

Again, we use the SPF to evaluate the zeta function, now termed $Z_3(t)$, for our new polynomial $f(z, v) = pz^2 - v^3$. Remember that the coefficient $p^{-1}t$ is not included in $Z_3(t)$.

Application 3

Now, to find $|\bar{N}|$ and $|\bar{S}|$, we consider the polynomial $\bar{f} = -v^3$ which is similar to the function we looked at before in the second application of SPF, and so by the same argument, $|\bar{N}| = 1$ and $|\bar{S}| = 1$. Using our formula for SPF:

$$\begin{aligned} Z_3(t) &= (p^n - |\bar{N}|)p^{-n} + (|\bar{N} - \bar{S}|)p^{-n}t\left(\frac{1-p^{-1}}{1-p^{-1}t}\right) + I_3 \\ &= (p-1)p^{-1} + (1-1)p^{-n}t\left(\frac{1-p^{-1}}{1-p^{-1}t}\right) + I_3 \\ &= (1-p^{-1}) + I_3 \end{aligned}$$

Using the substitution $v = pk$ and $dv = p^{-1}dk$, we can solve for I_3.

$$\begin{aligned} I_3 &= \int_{\mathbb{Z}_p}\int_{0+p\mathbb{Z}_p} |pz^2 - v^3|_p^s dvdz \\ &= p^{-1}\int_{\mathbb{Z}_p^2} |pz^2 - p^3k^3|_p^s dkdz \\ &= p^{-1}t\int_{\mathbb{Z}_p^2} |z^2 - p^2k^3|_p^s dkdz \\ &= p^{-1}tZ_4(t) \end{aligned}$$

Again we have an integral over $\mathbb{Z}_p^2$ which we will compute via SPF

Application 4

To find $Z_4(t)$, we consider the zeta function for the polynomial $f(z, k) = z^2 - p^2k^3$. So first looking at the function $\bar{f} = z^2$ we see it is analogous to the last case. So in this case too we have $|\bar{N}| = 1$ and $|\bar{S}| = 1$. Applying SPF once again we get:

$$\begin{aligned} Z_4(t) &= (p^n - |\bar{N}|)p^{-n} + (|\bar{N} - \bar{S}|)p^{-n}t\left(\frac{1-p^{-1}}{1-p^{-1}t}\right) + I_4 \\ &= (p-1)p^{-1} + (1-1)p^{-n}t\left(\frac{1-p^{-1}}{1-p^{-1}t}\right) + I_4 \\ &= (1-p^{-1}) + I_4 \end{aligned}$$

Using the substitution $z = pa$ and $dz = p^{-1}a$, we solve for I_4.

$$\begin{aligned} I_4 &= \int_{\mathbb{Z}_p}\int_{0+p\mathbb{Z}_p} |z^2 - p^2k^3|_p^s dzdk \\ &= p^{-1}\int_{\mathbb{Z}_p^2} |p^2a^2 - p^2k^3|_p^s dadk \end{aligned}$$

$$\begin{aligned}
&= p^{-1}t^2\int_{\mathbb{Z}_p^2}|a^2-k^3|_p^s da dk\\
&= p^{-1}t^2 Z(t)
\end{aligned}$$

We now have our original zeta function back. So the next step is to put all our different pieces together and solve for the original zeta function, and simplify. Starting from the beginning and substituting for the value we got for each application, we get:

$$\begin{aligned}
Z(t) &= (1-p^{-1})+(p-1)p^{-2}t\left(\frac{1-p^{-1}}{1-p^{-1}t}\right)\\
&\quad + p^{-2}t^2\left[(1-p^{-1})+p^{-1}t\left[(1-p^{-1})+p^{-1}t\left[(1-p^{-1})+p^{-1}t^2Z(t)\right]\right]\right]\\
&= (1-p^{-1})+(p-1)p^{-2}t\left(\frac{1-p^{-1}}{1-p^{-1}t}\right)\\
&\quad + p^{-2}t^2-p^{-3}t^2+p^{-3}t^3-p^{-4}t^3+p^{-4}t^4-p^{-5}t^4+p^{-5}t^6Z(t)\\
Z(t) &= \frac{1-p^{-1}-p^{-2}t+p^{-3}t+p^{-2}t^2-p^{-3}t^2-p^{-5}t^5+p^{-6}t^5}{(1-p^{-1}t)(1-p^{-5}t^6)}\\
&= \frac{(1-p^{-2}t+p^{-2}t^2-p^{-5}t^5)-p^{-1}(1-p^{-2}t+p^{-2}t^2-p^{-5}t^5)}{(1-p^{-1}t)(1-p^{-5}t^6)}\\
&= \frac{(1-p^{-1})(1-p^{-2}t+p^{-2}t^2-p^{-5}t^5)}{(1-p^{-1}t)(1-p^{-5}t^6)}
\end{aligned}$$

Therefore for the polynomial $f(x,y)=x^2-y^3$, $Z(t)$ is the Igusa local zeta function for all primes.

4. NATURAL BOUNDARY OF THE EULER PRODUCT OF THE IGUSA LOCAL ZETA FUNCTION OF $f(x,y)=x^2-y^3$

There exist functions analytic in a domain D which cannot be analytically continued to any point outside D. In such a case the boundary of D is called the natural boundary [?]. We want to find the natural boundary for the Euler product of the Igusa local zeta function of $f(x,y)=x^2-y^3$. Recalling from the last section, the Igusa zeta function for each prime p is given by:

$$Z(s)=\frac{(1-p^{-1})(1-p^{-2-s}+p^{-2-2s}-p^{-5-5s})}{(1-p^{-1-s})(1-p^{-5-6s})}$$

where $s=\sigma+i\tau$ and $t=p^{-s}$. We normalize the measure involved in the calculation $Z(t)$ by dividing by $1-p^{-1}$ for each p so that our local zeta factor is:

$$Z(s)=\frac{(1-p^{-2-s}+p^{-2-2s}-p^{-5-5s})}{(1-p^{-1-s})(1-p^{-5-6s})}$$

With this new $Z(t)$, we form the Euler product

$$\tilde{Z}(s)=\prod_p\frac{(1-p^{-2-s}+p^{-2-2s}-p^{-5-5s})}{(1-p^{-1-s})(1-p^{-5-6s})}$$

Our main question is to find the natural boundary for $\tilde{Z}(s)$. We know that:

$$\prod_p \frac{1}{1-p^{-1-s}} = \zeta(s+1)$$

and

$$\prod_p \frac{1}{1-p^{-5-6s}} = \zeta(6s+5)$$

Hence,

$$\tilde{Z}(s) = \left(\prod_p (1-p^{-2-s}+p^{-2-2s}-p^{-5-5s})\right)\zeta(s+1)\zeta(6s+5)$$

The Riemann zeta functions can be meromorphically continued on the entire complex plane, thus now our problem is to find the natural boundary of

$$F(s) = \prod_p (1-p^{-2-s}+p^{-2-2s}-p^{-5-5s})$$

We know that an infinite product $\prod_{n\in I}(1+a_n)$ converges absolutely if the corresponding sum $\sum_{n\in I}|a_n|$ converges, where I is an infinite index set. Now $\sum_{p \text{ prime}}|p^{-s}|$ converges on $\{s\in\mathbb{C} : Re(s) > 1\}$. Hence we see that in our infinite product $F(s)$ it is the term p^{-2-2s} that is the limit of convergence. Therefore $F(s)$ converges on $\{s\in\mathbb{C} : Re(s) > -\frac{1}{2}\}$.

We can see why that term is the limit of convergence if we take a look at each term in the numerator individually. $\sum_{p \text{ prime}}|p^{-(2+s)}|$ converges on $\{s\in\mathbb{C} : Re(s) > -1\}$. Also $\sum_{p \text{ prime}}|p^{-(2+2s)}|$ converges on $\{s\in\mathbb{C} : Re(s) > -\frac{1}{2}\}$ and $\sum_{p \text{ prime}}|p^{-(5+5s)}|$ converges on $\{s\in\mathbb{C} : Re(s) > -\frac{4}{5}\}$.

Our next step is to try to analytically continue the function to a larger region. Our technique [?] to continue this function, is to multiply with Riemann Zeta functions, which we know are meromorphic on the whole complex plane. The Riemann Zeta functions that we have multiplied top and bottom with, have poles at the same points. Let

$$H(s) = \prod_p (1-p^{-2-s}+p^{-2-2s}-p^{-5-5s})\frac{(1-p^{-2-2s})}{(1-p^{-4-4s})}$$

So in this case $H(s) = F(s).\frac{\zeta(4+4s)}{\zeta(2+2s)}$.Therefore if we can analytically continue $H(s)$, that means that we are in turn finding an analytic continuation for $F(s)$. The terms in the infinite product $H(s)$ can be written as

$$\begin{aligned} & \prod\nolimits_p \quad (1-p^{-2-s}+p^{-2-2s}-p^{-5-5s})\frac{(1-p^{-2-2s})}{(1-p^{-4-4s})} \\ = & \prod\nolimits_p \quad \frac{(1-p^{-2-s}+p^{-2-2s}-p^{-5-5s})}{(1+p^{-2-2s})} \\ = & \prod\nolimits_p \quad 1+\frac{(-p^{-2-s}-p^{-5-5s})}{(1+p^{-2-2s})} \end{aligned}$$

To find the radius of convergence of the infinite product of these terms we again look at the absolute convergence of sums. To determine the limit of convergence we look at the two sums. The first sum is:

$$\sum_p \frac{p^{-2-s}}{1+p^{-2-2s}} = \sum_p \frac{1}{p^{2+s}+p^{-s}} = \sum_p \frac{p^s}{p^{2+2s}+1}$$

Now we look at the limit of absolute convergence for this term, where $s=\sigma+i\tau$. Firstly we know that $|p^s|=p^\sigma$ so our next step is to find $|p^{2+2s}+1|$. Because of the triangle inequality we know that $|x|-|y|\leq|x-y|$, so applying it to our case we get that $p^{2+2\sigma}-1\leq|p^{2+2s}+1|$.And so we get:

$$\sum_p \left|\frac{p^s}{p^{2+2s}+1}\right| \leq \sum_p \frac{p^\sigma}{p^{2+2\sigma}-1} = \sum_p \frac{1}{p^{2+\sigma}-p^\sigma}$$

We use the limit comparison test to determine the limit of convergence. $\sum_p \frac{1}{p^{2+\sigma}}$ converges for $Re(s)>-1$, so we compare $\sum_p \frac{1}{p^{2+\sigma}-p^\sigma}$ and $\sum_p \frac{1}{p^{2+\sigma}}$. We get

$$\lim_{p\to\infty} \frac{p^{2+\sigma}}{p^{2+\sigma}-p^\sigma} = \lim_{p\to\infty} \frac{1}{1-p^{-2-2\sigma}}$$

which is 1 for $Re(s)>-1$. Therefore it means that$\sum_p \frac{1}{p^{2+\sigma}-p^\sigma}$also converges for $Re(s)>-1$. Our next step is to look at the second term.

$$\sum_p \frac{p^{-5-5s}}{1+p^{-2-2s}} = \sum_p \frac{1}{p^{5+5s}+p^{3+3s}} = \sum_p \frac{p^{-3-3s}}{p^{2+2s}+1}$$

Looking at the limit of absolute convergence for this term we see:

$$\sum_p \left|\frac{p^{-3-3s}}{p^{2+2s}+1}\right| \leq \sum_p \frac{p^{-3-3\sigma}}{p^{2+2\sigma}-1} = \sum_p \frac{1}{p^{5+5\sigma}-p^{3+3\sigma}}$$

Once again we use the limit comparison test and this time we compare our term to $\sum_p \frac{1}{p^{5+5\sigma}}$ which we know converges for $Re(s)>-\frac{4}{5}$. So we have

$$\lim_{p\to\infty} \frac{p^{5+5\sigma}}{p^{5+5\sigma}-p^{3+3\sigma}} = \lim_{p\to\infty} \frac{1}{1-p^{-2-2a}}$$

which is also equal to 1 for $Re(s)>-\frac{4}{5}$. Using the same argument as before, we know that $\sum_p \frac{p^{-3-3\sigma}}{p^{2+2\sigma}-1}$ converges for $Re(s)>-\frac{4}{5}$.

And so by this process we have analytically continued $F(s)$ to $\{s\in\mathbb{C}: Re(s)>-\frac{4}{5}\}$. Once again we look at the numerator term of $Z(s)$ with complex exponents, which is :

$$1-p^{-2-s}+p^{-2-2s}-p^{-5-5s}$$

Our next step is to find a boundary which is a limit point of zeros from $\{\ s\in\mathbb{C}: Re(s)>4\}$, and this determines our natural boundary. To do this we consider solutions of the equation $1-X^2Y+X^2Y^2-X^5Y^5=0$. We will be interested in the value of s for solutions of the form $(X,Y)=(p^{-1},p^{-s})$. We make the following substitution: $U=X^{\frac{4}{5}}Y$ and $V=X^{\frac{1}{5}}$ and consider the equation

$$F(U,V)=1-UV^6+U^2V^2-U^5V^5=0$$

This has a trivial solution at $(U,V)=(1,1)$. The partial derivatives at this point are given by

$$F_u(U,V)|_{(1,1)}=-V^6+2UV^2-5U^4V^5=-4$$

$$F_v(U,V)|_{(1,1)} = -6UV^5 + 2U^2V - 5U^5V^4 = -9$$

We can now use the Implicit Function Theorem to expand U as a function of V in the neighborhood around the solution$(1,1)$ to get

$$(U-1) = -\frac{9}{4}(V-1) + \Omega(V-1)$$

$$U = \frac{13}{4} - \frac{9}{4}V + \Omega(V-1)$$

where $\Omega(V-1)$ is a power series in $V-1$. So for p large enough at the point $V = p^{-\frac{1}{5}}$ and $U = p^{-\frac{4}{5}-s}$ we get a solution of $1 - p^{-2-s} + p^{-2-2s} - p^{-5-5s} = 0$ for values of s satisfying

$$p^{-\frac{4}{5}-s} = \frac{13}{4} - \frac{9}{4}p^{-\frac{1}{5}} + \Omega(p^{-\frac{1}{5}} - 1)$$

$$p^{-\frac{4}{5}-s} = \frac{13}{4} - \frac{9}{4}p^{-\frac{1}{5}} + \Omega(p^{-\frac{1}{5}} - 1).e^{2\pi ni}$$

Taking log on both sides and solving for s we get

$$s = -\frac{4}{5} - \frac{log(\frac{13}{4} - \frac{9}{4}p^{-\frac{1}{5}} + \Omega(p^{-\frac{1}{5}} - 1))}{logp} - \frac{2\pi ni}{logp}$$

for all $n \in \mathbb{Z}$. Now

$$\delta_p = -\frac{log(\frac{13}{4} - \frac{9}{4}p^{-\frac{1}{5}} + \Omega(p^{-\frac{1}{5}} - 1))}{logp} \to 0$$

as $p \to \infty$. This can be easily seen if we use L'Hopital's rule to find this limit. We know that

$$\lim_{p\to\infty} \frac{1}{logp} = 0$$

and the term we have in the numerator is basically finite since $\Omega(p^{-\frac{1}{5}} - 1)$ is a power series of a real number between -1 and 0, and so the partial derivatives disappear and we are left with a finite number of terms. If we fix some point $A = -\frac{4}{5} + ai$ on the boundary $Re(s) = -\frac{4}{5}$ then it is possible to arrange a sequence of integers n_p for each prime p such that

$$\frac{(2n_p - 1)\pi}{logp} \to a$$

as $p \to \infty$. Therefore every point A on the boundary is a limit point of zeros. Finally we must check that the zeros on the right hand side of this boundary. This follows since $\delta_p > 0$ for large enough p, since the numerator has a log of a number between 0 and 1 for a large p. So $Re(s) = -\frac{4}{5}$ is a natural boundary for $F(s)$ and it can not be analytically continued beyond that.

5. Conclusion

In the course of our study using the numerator, we found the natural boundary of the Euler product of a Igusa local zeta function. As a next step, we are interested in finding a general method to determine natural boundaries for any given Igusa local zeta function. Most importantly, we want to find out if the method proposed by Marcus du Sautoy in [?] works for all zeta functions. Furthermore we might also want to explore the possibility of a correlation between the natural boundaries and singularities of the Igusa local zeta functions.

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