A Stationary Phase Formula for Generalized Exponential Sums

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Abstract

This paper gives a formula for Generalized Exponential sums which depends entirely upon the singularities mod p, similar to the purpose of the Stationary Phase Formula for Igusa Local Zeta Functions. This new formula is then applied to two examples: strongly non-degenerate homogeneous polynomials and general quadratic polynomials. This work was completed as part of the Mount Holyoke Summer Mathematics Institute, an NSF funded REU Program. [1]

1 Introduction

In 1965, A. Weil introduced the Generalized Exponential Sum, $F^*(i^*)$, defined as

$$F^*(i^*) = \int_{\mathbb{Z}_p^n} \Psi(i^* f(x))dx$$

For the purposes of this paper, $\Psi(x)$ will be defined such that

$$\Psi(x) = e^{2\pi i (\text{rational part of } x)}$$

In 1975, Jun-ichi Igusa proposed the Igusa Local Zeta Function, defined as

$$Z(t) = \int_{\mathbb{Z}_p^n} |f(x)|_p^s dx$$

where $f(x) \in \mathbb{Z}_p[x_1, x_2 ..., x_n]$. As is noted on page 1 of [Den],

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"Igusa's local zeta function is directly related to the number of solutions of the congruences $f(x) \equiv 0(\text{mod} p^m)$, $m = 1, 2, 3, ...$". It is interesting to note that Igusa found (also in 1975) that this integral could be expressed as a rational function of p^{-1} and p^s. More importantly for the purposes of this paper, however, Igusa found that there exists a one-to-one mapping between Igusa Local Zeta Functions and Generalized Exponential Sums; if one is given a Local Zeta Function, then that person could use an inverse Mellon Transform and a Fourier Transform to map this Zeta Function to a unique Generalized Exponential Sum, while the inverse could be done to map a Generalized Exponential Sum back to a unique Zeta Function.

It was discovered in 1994 by Igusa in [Igu94] that there exists an organizing method, known as the Stationary Phase Formula (or SPF), which established that the form of an Igusa Local Zeta Function was dictated by the singular points (the points where all of the partial derivatives evaluated to zero) modulo p. The formula is expressed as follows:

$$Z(t) = (p^n - |\overline{N}|)p^{-n} + (|\overline{N}| - |\overline{S}|)p^{-n}t(\tfrac{1-p^{-1}}{1-p^{-1}t}) + \Sigma_{a \in S} \int_{a+p\mathbb{Z}_p} |f(x)|_p^s dx$$

where n is the number of variables in $f(x)$, $\overline{N}$ is the set of points a such that $f(a) \equiv 0(\text{mod} p)$, $\overline{S}$ is the set of points in $\overline{N}$ where $\frac{\partial f}{\partial x_i}(a) \equiv 0(\text{mod} p)\ \forall\ i$, and $t = p^{-s}$.

Because of the relationship between Igusa Local Zeta Functions and Generalized Exponential Sums, it seemed a valid question to ask whether the Exponential Sums had a similar organizing principle. In this paper, it is found that the answer is yes, although the exact definition of a singular point is broadened to include points where the partial derivatives all evaluate to zero modulo p but the function itself evaluates to a unit modulo p. Section 2 describes this result as well as its proof.

In Section 3, we begin to apply the result in Section 2 some examples. The first of these examples is the case of the strongly nondegenerate homogeneous polynomial (i.e. a polynomial whose only singular point is at (0,0,..,0)). The result for this example appears in [Igu00], although the proof is left as an exercise for the reader. In Section 4, we turn our attention to general quadratic polynomials, discovering some interesting results about which singular points we can ignore in our evaluation of $F^*(i^*)$.

2 The Formula

Theorem 2.1: Let

$$F^*(i^*) = \int_{\mathbb{Z}_p^n} \Psi(i^* f(x_1, x_2, ..., x_n))dx_1 dx_2 ... dx_n$$

where $i^* = p^{-e}u$. Further, let a be a vector such that $a \in \mathbb{F}_p^n$. We define S such that $a \in S$ if $\frac{\partial f}{\partial x_i}(a) \equiv 0 \pmod{p}\ \forall\ i$ (i.e $a \in S$ if a is a singular point mod p). If $e > 1$ then

$$F^*(i^*) = \Sigma_{a \in S} \int_{a+p\mathbb{Z}_p^n} \Psi(i^* f(x_1, x_2, ..., x_n))dx_1 dx_2 ... dx_n.$$

Proof: We can break $F^*(i^*)$ into a sum of integrals of the vectors a which are not singular points mod p plus a sum of integrals over the vectors a which are singular points mod p as follows:

$$F^*(i^*) = \Sigma_{a \notin S} \int_{a+p\mathbb{Z}_p^n} \Psi(i^* f(x_1, x_2, ..., x_n))dx_1 dx_2 ... dx_n$$
$$+\Sigma_{a \in S} \int_{a+p\mathbb{Z}_p^n} \Psi(i^* f(x_1, x_2, ..., x_n))dx_1 dx_2 ... dx_n$$

Now, we must show that the first integral is zero. By a change of variables

$$\Sigma_{a \notin S} \int_{a+p\mathbb{Z}_p^n} \Psi(i^* f(x_1, x_2, ..., x_n))dx_1 dx_2 ... dx_n$$
$$= p^{-n} \Sigma_{a \notin S} \int_{\mathbb{Z}_p} \Psi(i^* f(a_1 + px_1, a_2 + px_2, ..., a_n + px_n))dx_1 dx_2 ... dx_n$$

which, by Taylor's Theorem, is equal to

$$p^{-n} \Sigma_{a \notin S} \int_{\mathbb{Z}_p} \Psi((p^{-e}u)(f(a_1, a_2, ..., a_n) + \Sigma p x_i \frac{\partial f}{\partial x_i}(a) + p^2(...)))dx_1 dx_2 ... dx_n$$
$$= p^{-n} \Sigma_{a \notin S} \Psi((p^{-e}u) f(a_1, a_2, ..., a_n)) \int_{\mathbb{Z}_p} \Psi((p^{-e}u) \Sigma p x_i \frac{\partial f}{\partial x_i}(a) + p^2(...))dx_1 dx_2 ... dx_n$$

since Ψ is an additive character. By assumption, one of the partial derivatives must be congruent to a unit modulo p. So let us define $\frac{\partial f}{\partial x_j}(a) = c_j\ \forall\ j$ and let c_i be a partial which is a unit mod p for some i. We change variables as follows:

$x'_j = x_j \ \forall\ j \neq i$
$x'_i = \Sigma_{j=o}^{n} c_j x_j + p(...)$

This can be shown to be measure invariant through an argument used first by Jun-Ichi Igusa in [Igu94], although the argument presented here comes from [Fle]. Let y be the vector $(x_1, x_2, ..., x_n)$ and let y' be the vector $(x'_1, x'_2, ..., x'_n)$, which means that $dy = dx_1 dx_2 ... dx_n$ and $dy' = dx'_1 dx'_2 ... dx'_n$. So:

$$y' = \begin{pmatrix} x'_1 \\ x'_2 \\ ... \\ x'_i \\ ... \\ x'_n \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ ... \\ c_1 x_1 + c_2 x_2 + ... c_n x_n + p(...) \\ ... \\ x_n \end{pmatrix}$$

This implies that

$$\frac{\partial(x'_1, x'_2, ..., x'_n)}{\partial(x_1, x_2, ..., x_n)} = \begin{vmatrix} 1 & 0 & ... & 0 & ... & 0 \\ 0 & 1 & ... & 0 & ... & 0 \\ ... & ... & ... & ... & ... & ... \\ c_1 + p(...) & c_2 + p(...) & ... & c_i + p(...) & ... & c_n + p(...) \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & ... & 0 & ... & 1 \end{vmatrix}$$
$$= c_i + p(...)$$

This can be used to compute the change in measure from dy to dy':

$$dy' = \left| \frac{\partial(x'_1, x'_2, ..., x'_n)}{\partial(x_1, x_2, ..., x_n)} \right|_p dy$$
$$dy' = |c_i + p(...)|_p \, dy$$
$$dy' = dy$$

Thus, the change of variables is measure invariant. Using this change of variables,

$$\int_{\mathbb{Z}_p} \Psi((p^{-e}u) \Sigma p x_i \tfrac{\partial f}{\partial x_i}(a) + p^2(...)) dx_1 dx_2 ... dx_n = \int_{\mathbb{Z}_p} \Psi((p^{-e+1}u) x'_i) dx'_1 dx'_2 ... dx'_n$$
$$= \int_{\mathbb{Z}_p} \Psi(p^{-e+1} u x'_i) dx'_i$$

Clearly,

$$\int_{\mathbb{Z}_p} \Psi(p^{-e+1}ux_i)dx_i = 0$$

and thus

$$\Sigma_{a\notin S} \int_{a+p\mathbb{Z}_p^n} \Psi(i^*f(x_1, x_2, ..., x_n))dx_1dx_2...dx_n = 0. \boxplus$$

2.1 Remark

Unlike the Stationary Phase Formula for Igusa Local Zeta Functions, the SPF for Weil's function includes in S the points where the partial derivatives evaluate to zero modulo p but the function itself does not. This is done because these particular singularities do not always evaluate to zero in the same manner as the non-singular points. For example, let us consider a function where none of the singular points are places where the function evaluates to zero modulo p:

$$F^*(i^*) = \int_{\mathbb{Z}_p} \Psi(p^{-e}u(x^2 + x))dx$$

where e=2, p=3. We can break the integral into a sum of integrals as we did in section 1:

$$F^*(i^*) = \Sigma_{a\notin S} \int_{a+p\mathbb{Z}_p} \Psi(p^{-e}u(x^2 + x))dx$$
$$+\Sigma_{a\in S} \int_{a+p\mathbb{Z}_p} \Psi(p^{-e}u(x^2 + x))dx$$

We can see that the singularities mod p are $S = \{1\}$ and we know that the first integral vanishes. So

$$F^*(i^*) = \Sigma_{a=1} \int_{a+p\mathbb{Z}_p} \Psi(3^{-2}u(x^2 + x))dx$$
$$= p^{-1}\Sigma_{a=1} \int_{\mathbb{Z}_p} \Psi(3^{-2}u(f(a) + f'(a)px + f''(a)p^2x^2))dx$$

since all derivatives after the second are zero. Further, because $a \in S$, $f'(a) = 0$. This implies that

$$F^*(i^*) = p^{-1}\Psi(3^{-2}u(f(a))) \int_{\mathbb{Z}_p} \Psi(3^{-2}u(f''(a)p^2x^2))dx$$
$$= p^{-1}\Psi(3^{-2}u(2)) \int_{\mathbb{Z}_p} \Psi(u(2x^2))dx$$
$$= p^{-1}\Psi(3^{-2}u(2)) = \frac{1}{3}e^{\frac{4\pi i}{9}}$$

Note that although none of the singular points were places where the function evaluated to zero modulo p, we still had to deal with these points individually, as the integrals over the balls with these centers did not evaluate to zero.

3 An Example: the General Homogeneous Equation

Let $f(x_1, x_2, ..., x_n)$ be a homogeneous polynomial of degree k where the only singular point is at $a = (0, 0, ..., 0)$. Again, let

$$F^*(i^*) = \int_{\mathbb{Z}_p^n} \Psi(i^* f(x_1, x_2, ..., x_n)) dx_1 dx_2 ... dx_n$$

where $i^* = p^{-e}u$. Express e as $ck + d$ for some $c, d \in \mathbb{Z}$ such that $c \geq 0$ and $-k + 2 \leq d \leq 1$. If $e = 1$ then we simply evaluate the integral

$$F^*(i^*) = \int_{\mathbb{Z}_p^n} \Psi(i^* f(x_1, x_2, ..., x_n)) dx_1 dx_2 ... dx_n$$
$$= \Sigma_{t \in \mathbb{Z} \bmod p} (t/p) |N_1(t)|$$

where $N_1(t)$ is the set of points such that $f(x_1, x_2, ..., x_n)$ evaluates to $t \pmod p$. Otherwise,

$$F^*(i^*) = \int_{\mathbb{Z}_p^n} \Psi(i^* f(x_1, x_2, ..., x_n)) dx_1 dx_2 ... dx_n$$
$$= \Sigma_{a \in S} \int_{a + p\mathbb{Z}_p^n} \Psi(p^{-e} u f(x_1, x_2, ..., x_n)) dx_1 dx_2 ... dx_n$$
$$= \int_{p\mathbb{Z}_p^n} \Psi(p^{-e} u f(x_1, x_2, ..., x_n)) dx_1 dx_2 ... dx_n$$
$$= p^{-n} \int_{\mathbb{Z}_p^n} \Psi(p^{-e} u f(px_1, px_2, ..., px_n)) dx_1 dx_2 ... dx_n$$
$$= p^{-n} \int_{\mathbb{Z}_p^n} \Psi(p^{-e+k} u f(x_1, x_2, ..., x_n)) dx_1 dx_2 ... dx_n$$

by change of variables. This change of variables can be performed c times to find that

$$F^*(i^*) = p^{-cn} \int_{\mathbb{Z}_p^n} \Psi(p^{-e+ck} u f(x_1, x_2, ..., x_n)) dx_1 dx_2 ... dx_n$$
$$= p^{-cn} \int_{\mathbb{Z}_p^n} \Psi(p^{-d} u f(x_1, x_2, ..., x_n)) dx_1 dx_2 ... dx_n$$

If $d = 1$ (i.e. $e \equiv 1 \pmod{k}$), we evaluate the integral

$$
\begin{aligned}
&p^{-cn} \int_{\mathbb{Z}_p^n} \Psi(p^{-d} u f(x_1, x_2, ..., x_n)) dx_1 dx_2 ... dx_n \\
&= p^{-cn} \int_{\mathbb{Z}_p^n} \Psi(p^{-1} u f(x_1, x_2, ..., x_n)) dx_1 dx_2 ... dx_n \\
&= p^{-\frac{e-1}{k} n} \Sigma_{t \in \mathbb{Z} \bmod p} (t/p) |N_1(t)|
\end{aligned}
$$

If $d \leq 0$, we find that

$$p^{-cn} \int_{\mathbb{Z}_p^n} \Psi(p^{-d} u f(x_1, x_2, ..., x_n)) dx_1 dx_2 ... dx_n = p^{-cn} = p^{n[-\frac{e}{k}]}$$

since the psi function of an integer is 1 and $-c = [-\frac{e}{k}]$ by definition of c. Thus, for the general homogeneous equation,

$$F^*(i^*) = \begin{cases} p^{n[-\frac{e}{k}]} & \text{if } e \not\equiv 1 (\text{mod } k) \\ p^{-\frac{e-1}{k} n} \sum_{t \in \mathbb{F}_p^n} (\frac{t}{p}) |N_1(t)| & \text{if } e \equiv 1 (\text{mod } k) \end{cases}$$

4 Another Example: the General Quadratic Equation

4.1 Lemmas

Now, we prove a lemma about the singular points of the general quadratic equation:

Lemma 4.1.1: If $f(x, y) = b_1 x^2 + b_2 xy + b_3 y^2$ has singular points other than $a = (0, 0)$ then $b_2^2 - 4b_1 b_3 \equiv 0 \pmod{p}$

Proof: We know that if a is a singular point then

$2b_1 x + b_2 y \equiv 0 \pmod{p}$ and
$b_2 x + 2b_3 y \equiv 0 \pmod{p}$

Multiplying the first equation by $2b_3$ and the second by b_2 and then subtracting one from the other,

$(4b_1 b_3 - b_2^2)x \equiv 0 \pmod{p}$

Now, if $x \equiv 0$ then $y \equiv 0$. So assume $x \not\equiv 0$. Then

$$4b_1b_3 - b_2^2 \equiv 0 \pmod{p}. \boxplus$$

Lemma 4.1.2: If there exist singular points for the equation

$$rx + sy + b_1x^2 + b_2xy + b_3y^2 + p(...)$$

then

$$\frac{-r}{2b_1} \equiv -\frac{s}{b_2} \pmod{p}$$

Proof: If (u, v) is a singular point for the equation

$$rx + sy + b_1x^2 + b_2xy + b_3y^2 + p(...)$$

then

$$r + 2b_1u + b_2v \equiv 0 \pmod{p} \text{ and}$$
$$s + b_2u + 2b_3v \equiv 0 \pmod{p}$$

By Lemma 4.1.1, $b_2^2 \equiv 4b_1b_3 \pmod{p}$. So, multipling the second equation by $2b_1$ and substituting b_2^2 for $4b_1b_3$,

$$r + 2b_1u + b_2v \equiv 0 \pmod{p} \text{ and}$$
$$2b_1s + 2b_1b_2u + b_2^2v \equiv 0 \pmod{p}$$

Solving for u in terms of v,

$$u \equiv \frac{-r-b_2v}{2b_1} \pmod{p} \text{ and}$$
$$u \equiv -\frac{s}{b_2} - \frac{b_2v}{2b_1} \pmod{p}$$

Seting the expressions for u to be congruent to one another,

$$\frac{-r-b_2v}{2b_1} = -\frac{s}{b_2} - \frac{b_2v}{2b_1} \pmod{p}$$

or, equivalently,

$$\frac{-r}{2b_1} \equiv -\frac{s}{b_2} \pmod{p}. \boxplus$$

Using these lemmas, we prove that the Generalized Exponential Sum does not examine certain singular points for the general quadratic equation.

Theorem 4.2.1: Let $f_0(x,y) = b_1x^2 + b_2xy + b_3y^2$. Further, $\forall$ a, let h_a be the maximum power of p such that $\forall$ i, $\frac{\partial f}{\partial x_i}(a) \equiv 0 (\text{mod} p^{h_a})$ for that a. Let $h_{a'} = min\{h_a | a \in S\}$ and let $a \in S'$ if $h_{a'} = h_a$ for that particular a and $b_2\frac{\partial f}{\partial x}(a) \not\equiv 2b_1\frac{\partial f}{\partial y}(a)(\text{mod} p^{h_{a'}+1})$. Assume that $h_{a'} < \frac{e}{2}$. Then

$$F^*(i^*) = \Sigma_{a\in S, a\notin S'} \int_{a+p\mathbb{Z}_p^n} \Psi(p^{-e}u(b_1x^2 + b_2xy + b_3y^2))dxdy$$

Proof: By the same change of variables as in Theorem 2.1,

$$F^*(i^*) = p^{-2}\Sigma_{a\in S} \int_{\mathbb{Z}_p^n} \Psi(p^{-e}u(f(a) + p\frac{\partial f}{\partial x'}(a)x' + p\frac{\partial f}{\partial y'}(a)y' \\ + p^2(b_1x'^2 + b_2x'y' + b_3y'^2)))dx'dy'$$

So we must show that

$$p^{-2}\Sigma_{a\in S, a\in S'} \int_{\mathbb{Z}_p^n} \Psi(p^{-e}u(f(a) + p\frac{\partial f}{\partial x'}(a)x' + p\frac{\partial f}{\partial y'}(a)y' \\ + p^2(b_1x'^2 + b_2x'y' + b_3y'^2)))dx'dy' = 0$$

Now, by definition of h_a,

$$\frac{\partial f}{\partial x'}(a)x' + \frac{\partial f}{\partial y'}(a)y' = p^{h_a}(r_ax' + s_ay')$$

where at least one of r_a, s_a is a unit. Further, if $a \in S'$ then

$$\frac{\partial f}{\partial x'}(a)x' + \frac{\partial f}{\partial y'}(a)y' = p^{h_{a'}}(r_ax' + s_ay')$$

where $h_{a'}$ is the minimal h_a. So

$$p^{-2}\Sigma_{a\in S, a\in S'} \int_{\mathbb{Z}_p^n} \Psi(p^{-e}u(f(a) + p\frac{\partial f}{\partial x'}(a)x' + p\frac{\partial f}{\partial y'}(a)y' \\ + p^2(b_1x'^2 + b_2x'y' + b_3y'^2)))dx'dy' \\ = p^{-2}\Sigma_{a\in S, a\in S'} \int_{\mathbb{Z}_p^n} \Psi(p^{-e}u(f(a) + p^{h_{a'}}(r_ax' + s_ay')p \\ + p^2(b_1x'^2 + b_2x'y' + b_3y'^2)))dx'dy' \\ = p^{-2}\Sigma_{a\in S, a\in S'}\Psi(p^{-e}uf(a)) \int_{\mathbb{Z}_p^n} \Psi(p^{-e+2}u(p^{h_{a'}-1}(r_ax' + s_ay')$$

$$+(b_1x'^2 + b_2x'y' + b_3y'^2)))dx'dy'$$

since Psi is additive. Now let

$$f_1(x,y) = b_1x^2 + b_2xy + b_3y^2 + p^{h_a-1}(r_a x + s_a y)$$

So we can rewrite the integral as

$$p^{-2}\Sigma_{a\in S, a\in S'}\Psi(p^{-e}uf(a))\Sigma_{a_1\in S}\int_{a_1+p\mathbb{Z}_p^n}\Psi(p^{-e+2}uf_1(x',y'))dx'dy'$$

More generally, for some $j \in \mathbb{N}$ where $j < h_{a'}$, let

$$f_j(x,y) = b_1x^2+b_2xy+b_3y^2+p^{h_{a'}-j}(r_a x+s_a y)+p^{h_{a'}-j+1}(c_j(x,y))$$

for some polynomial $c_j(x,y) \in \mathbb{Z}_p[x,y]$. Since $f_j \equiv f_0 \pmod p$ implies that the two polynomials have the same singular points mod p,

$$\begin{aligned}&\int_{\mathbb{Z}_p^n}\Psi(p^{-e+2j}u(f_j(x',y'))dx'dy'\\ &= p^{-2}\Sigma_{a_j\in S}\Psi(p^{-e+j+1}u(f_i(a_j)))\int_{\mathbb{Z}_p^n}\Psi(p^{-e+j+2}u(p^{h_{a'}-2j}(r_a x'\\ &\quad +s_a y') + b_1x'^2 + b_2x'y' + b_3y'^2 + (p^{h_{a'}-2j+1}(c_{j+1}(x',y')))))dx'dy'\end{aligned}$$

for some $c_{j+1}(x,y) \in \mathbb{Z}_p[x,y]$. Note that the new integral can be expressed as

$$\int_{\mathbb{Z}_p^n}\Psi(p^{-e+j+2}u(p^{h_{a'}-2j}(f_{j+1}(x',y')))dx'dy'$$

where $f_{j+1}(x,y)$ is defined as $f_j(x,y)$ was previously. So $\forall\ j \le h_{a'}$,

$$\begin{aligned}&F^*(i^*) = p^{-2j}\Sigma_{a\in S,a\in S'}\Sigma_{a_1\in S}...\Sigma_{a_{j-1}\in S}\Psi(p^{-e}uf(a))(p^{-e+2}u(f_1(a_1)))...\\ &\quad ...\Psi(p^{-e+2j}u(f_{j-1}(a_{j-1})))\int_{\mathbb{Z}_p^n}\Psi(p^{-e+2j}u(f_j(x',y'))dx'dy'\end{aligned}$$

Further, if $j = h_{a'}$,

$$\begin{aligned}&F^*(i^*) = p^{-2h_{a'}}\Sigma_{a\in S,a\in S'}\Sigma_{a_1\in S}...\Sigma_{a_{h_{a'}-1}\in S}\Psi(p^{-e}uf(a))(p^{-e+2}u(f_1(a_1)))...\\ &\quad ...\Psi(p^{-e+2h_{a'}}u(f_{h_{a'}-1}(a_{h_{a'}-1})))\int_{\mathbb{Z}_p^n}\Psi(p^{-e+2h_{a'}}u((r_a x' + s_a y')\\ &\quad +(b_1x'^2 + b_2x'y' + b_3y'^2) + p(c_{h_{a'}}(x',y'))))dx'dy'\end{aligned}$$

By assumption, $b_2\frac{\partial f}{\partial x}(a) \not\equiv 2b_1\frac{\partial f}{\partial x}(a) \pmod{p^{h_{a'}+1}}$. But $r_a = \frac{\partial f}{\partial y}(a)$

and $s_a = \frac{\partial f}{\partial y}(a)$. So $\frac{-r_{a'}}{2b_1} \not\equiv -\frac{s_{a'}}{b_2} \pmod p$. But then by Lemma 4.1.1, $r_{a'}x' + s_{a'}y' + b_1x'^2 + b_2x'y' + b_3y'^2$ has no singular points mod p. By Theorem 2.1, this implies that this integral is zero. So

$$p^{-2}\Sigma_{a\in S, a\in S'} \int_{\mathbb{Z}_p^n} \Psi(p^{-e}u(f(a) + p\frac{\partial f}{\partial x'}(a)x' + p\frac{\partial f}{\partial y'}(a)y' + p^2(b_1x'^2 + b_2x'y' + b_3y'^2)))dx'dy' = 0. \boxplus$$

4.2 An Example from 4.1

Let $f(x, y) = 6x^2 + 3xy + y^2, p = 5$.

So the singular points are wherever $12x + 3y \equiv 0 \pmod p$ and $3x + 2y \equiv 0 \pmod p$, which means that (x, y) is a singular point iff $x \equiv y \pmod p$. This means that the possible $a \in \mathbb{F}_p^2$ which are singular points are $a = (0, 0), (1, 1), (2, 2), (3, 3), (4, 4)$. We note that for all of these except $(0, 0)$,

$$12x + 3y \not\equiv 3x + 2y \pmod{p^2}.$$

Thus,

$$\begin{aligned} &F^*(i^*) = \Sigma_{a\in S} \int_{a+p\mathbb{Z}_p^n} \Psi(p^{-e}u(b_1x^2 + b_2xy + b_3y^2))dxdy \\ &= \int_{(0,0)+p\mathbb{Z}_p^n} \Psi(p^{-e}u(b_1x^2 + b_2xy + b_3y^2))dxdy \\ &= p^{-2} \int_{\mathbb{Z}_p^n} \Psi(p^{-e+2}u(b_1x^2 + b_2xy + b_3y^2))dxdy \end{aligned}$$

Since this change of variables yields the same function for integration and the only singular point of concern is $(0, 0)$, the process is the same as in Section 3, which implies that

$$F^*(i^*) = \begin{cases} p^{-e} & \text{if } e \text{ is even} \\ p^{-e+1}\Sigma_{t\in\mathbb{Z} \bmod p}(t/p)|N_1(t)| & \text{if } e \text{ is odd} \end{cases}$$

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