COMPUTING IGUSA LOCAL ZETA FUNCTIONS USING A NEWTON POLYHEDRON METHOD FOR DEGENERATE POLYNOMIALS

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ABSTRACT. This paper provides a method for computing the Igusa local zeta function associated to a polynomial which is degenerate with respect to some of the faces of its Newton polyhedron. The paper includes several examples of the calculation of the Igusa local zeta functions for partially degenerate polynomials. It also provides an introduction to p-adic analysis and an introduction to methods used to compute Igusa local zeta functions.

1. INTRODUCTION

This paper provides a method for computing the Igusa local zeta function $Z(s)$ associated to a polynomial which is degenerate with respect to some faces of its Newton polyhedron. Several examples are computed explicitly in two and three dimensions. We used the computer programs *Polygusa* [5] by Kathleen Hoornaert and Davy Loots and *Zeros* [10] by Joanna Miles. In addition, the paper provides background material relating to p-adic numbers and p-adic analysis, including an introduction to the Stationary Phase Formula [7], a formula for computing $Z(s)$ not involving the Newton polyhedron. This work was done in 2005 at the Summer Research Institute in Mathematics at Mouth Holyoke College under the direction of Margaret Robinson.

2. p-ADIC NUMBERS

2.1. p-adic valuation.

The definitions in this section can also be found in Gouvea [3], and Koblitz. [9]

Definition 2.1. Given a number $a \in \mathbb{Q}$, the *p-adic absolute value of a*, denoted $|a|_p$, is defined as

$$|a|_p = \begin{cases} p^{-ord_p(a)} & \text{if } a \neq 0 \\ 0 & \text{if } a = 0. \end{cases}$$

The quantity $ord_p(a)$ is the highest order of p dividing a. For example, $ord_5(\frac{1}{25}) = -2$, $ord_3(18) = 2$, and $ord_7(23) = 0$, so $|\frac{1}{25}|_5 = 5^2 = 25$, $|18|_3 = 3^{-2} = \frac{1}{9}$, and $|23|_7 = 7^0 = 1$.

This work was supported by NSF grant #DMS-0353700.

Definition 2.2. A sequence $\{x_i\}$ is *Cauchy* if $\forall\, \epsilon > 0$, $\exists\, N \in \mathbb{N}$ s.t. if $m, n > N$ then $|x_n - x_m| < \epsilon$.

Two Cauchy sequences $\{a_n\}$ and $\{b_n\}$ are equivalent if $|a_i - b_i| \to 0$ as $i \to \infty$.

Definition 2.3. The field of all p-adic numbers, $\mathbb{Q}_p$, is defined as all equivalence classes of p-adic Cauchy sequences.

By adding the limits of all p-adic Cauchy sequences to $\mathbb{Q}$, we form $\mathbb{Q}_p$, the *completion* of $\mathbb{Q}$ with respect to the p-adic absolute value, just as $\mathbb{R}$ is the completion of $\mathbb{Q}$ with respect to the usual absolute value.

Every p-adic number has the form

$$p^{-m}a_{-m} + p^{-m+1}a_{-m+1} + \ldots + a_0 + pa_1 + \ldots + p^{m-1}a_{m-1} + p^m a_m + \ldots$$

for some $m \in \mathbb{Z}$, with $0 \leq a_i \leq p-1$. [3]

Definition 2.4. The ring of p-adic integers, $\mathbb{Z}_p \subset \mathbb{Q}_p$, is composed of all p-adic numbers a with $|a|_p \leq 1$.

Every p-adic integer is of the form $a_0 + pa_1 + \ldots + p^{m-1}a_{m-1} + p^m a_m + \ldots$ for some $m \in \mathbb{Z}$. [3] The units in $\mathbb{Z}_p$ are elements in $\mathbb{Z}_p$ with $a_0 \neq 0$ and are denoted $\mathbb{Z}_p \backslash p\mathbb{Z}_p$ or $\mathbb{Z}_p^*$.

2.2. **Metric and Topology.** The existence of the p-adic absolute value induces a metric and a topology, on the field $\mathbb{Q}_p$. For more information on basic p-adic integration, see [8].

Definition 2.5. The distance d between two elements $x, y \in \mathbb{Q}_p$ is defined as

$$d(x,y) = |x-y|_p.$$

Proposition 2.6. *Properties of the metric on* $\mathbb{Q}_p$*:*

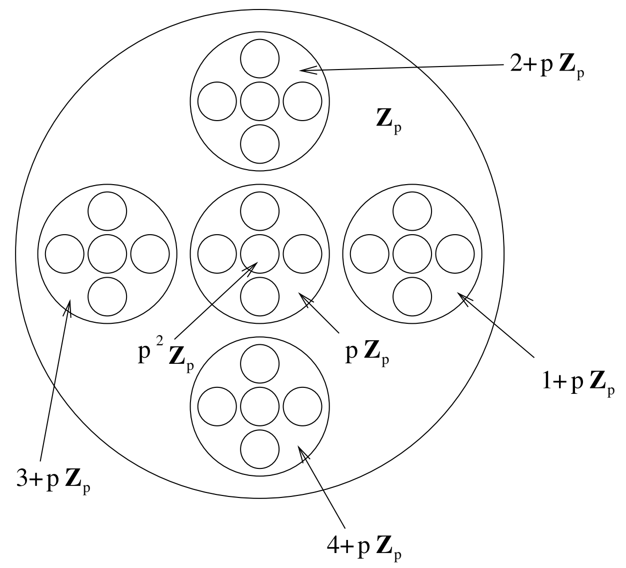
1. *For all* $x, y \in \mathbb{Q}_p$, $d(x,y) \geq 0$, *and* $d(x,y) = 0 \Leftrightarrow x = y$
2. *For all* $x, y \in \mathbb{Q}_p$, $d(x,y) = d(y,x)$
3. *For all* $x, y, z \in \mathbb{Q}_p$, $d(x,z) \leq d(x,y) + d(y,z)$
4. *For all* $x, y, z \in \mathbb{Q}_p$, $d(x,y) \leq \max\{d(x,z), d(z,y)\}$.

The first three of these properties form the definiton of a metric, in any field. A metric with the fourth property, called the *ultrametric property*, such as the p-adic absolute value, is called *non-archimedian.*

Since $\mathbb{Q}_p$ has a non-archimedian absolute value, its topology has several interesting properties.

1. Every point contained in an open (or closed) ball in $\mathbb{Q}_p$ is a center of that ball.
2. A ball in $\mathbb{Q}_p$ is both open and closed.
3. Any two open balls in $\mathbb{Q}_p$ are either disjoint or one is contained in the other.

The topology of $\mathbb{Z}_p$, a subring of $\mathbb{Q}_p$, for $p = 5$, is shown below.



2.3. **Measure on $\mathbb{Z}_p$.** $\mathbb{Z}_p$ is a locally compact topological group, so there exists a unique Haar measure on $\mathbb{Z}_p$ up to a positive real number. For more information on the Haar measure, see [1].

Proposition 2.7. *Properties of the Haar measure $m(E)$, $E = \alpha + p^e\mathbb{Z}_p$:*

1. $m(E) \geq 0$, $m(\emptyset) = 0$
2. *Given* E_1, E_2, $E_1 \cap E_2 = \emptyset$, $m(E_1 + E_2) = m(E_1) + m(E_2)$
3. $m(\beta + E) = m(E)$ *(translation invariance)*
4. $m(\mathbb{Z}_p) = 1.$

Example 2.8. Given that we normalize the measure of $\mathbb{Z}_p$ to be equal to 1, i.e.

$$m(\mathbb{Z}_p) = \int_{\mathbb{Z}_p} dx = 1,$$

we can show that

$$m(p\mathbb{Z}_p) = \int_{p\mathbb{Z}_p} dx = p^{-1}$$

as follows:
Since

$$\mathbb{Z}_p = \bigcup_{a=0}^{p-1} a + p\mathbb{Z}_p,$$

we know that

$$1 = \int_{\mathbb{Z}_p} dx = \sum_{a=0}^{p-1} \int_{a+p\mathbb{Z}_p} dx = p \int_{p\mathbb{Z}_p} dx,$$

so we see

$$\int_{p\mathbb{Z}_p} dx = p^{-1}.$$

Example 2.9.

$$\text{Since } \mathbb{Z}_p^* = \bigcup_{a=1}^{p-1} a + p\mathbb{Z}_p,$$

$$\begin{aligned} m(\mathbb{Z}_p \backslash p\mathbb{Z}_p) = m(\mathbb{Z}_p^*) &= \sum_{a=1}^{p-1} \int_{a+p\mathbb{Z}_p} dx \\ &= (p-1) \int_{p\mathbb{Z}_p} dx = (p-1)p^{-1} = 1 - p^{-1}. \end{aligned}$$

3. Igusa Local Zeta Function

Definition 3.1. The Igusa local zeta function associated to a polynomial $f(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$ is defined as

$$Z(s) = \int_{\mathbb{Z}_p^n} \mid f(x_1, \ldots, x_n) \mid_p^s dx_1 \ldots dx_n,$$

with $s \in \mathbb{C}$, $Re(s) > 0$.

Remark 3.2. We use the convention $t = p^{-s}$ throughout this paper.

3.1. **Stationary Phase Formula.** J.-I. Igusa in 1975 proved that all Igusa local zeta functions are rational functions of t [6] using the fact proved by H. Hironaka that a resolution of singularities exists for all polynomials over a field of characteristic zero. [4] One method for computing the Igusa local zeta function associated to a polynomial, the Stationary Phase Formula, was introduced by Igusa in 1994. [7]

Theorem 3.3. *Stationary Phase Formula:*

$$Z(s) = (p^n - |N_0|)p^{-n} + (|N_0| - |S|)p^{-n}t \left(\frac{1 - p^{-1}}{1 - p^{-1}t} \right)$$

$$+ \sum_{\alpha \in S} \int_{\alpha + p\mathbb{Z}_p^n} \mid f(x_1, \ldots, x_n) \mid^s dx_1 \ldots dx_n$$

where $N_0 = \{(x_1, \ldots, x_n) \in \mathbb{F}_p^n \mid f(x_1, \ldots, x_n) \equiv 0 \pmod{p}\}$
and $S = \{\bar{x} = (x_1, \ldots, x_n) \in N_0 \mid \frac{\partial f}{\partial x_i}(\bar{x}) \equiv 0 \pmod{p},\ 1 \leq i \leq n\}$.

Example 3.4. We will compute $Z(s)$ for the polynomial $f(x) = x$ using SPF. The first step is to compute $|N_0|$ and $|S|$ for $f(x)$.

$$N_0 = \{x \in \mathbb{F}_p \mid x \equiv 0 \pmod p\} = \{0\}$$

and

$$S = \{x \in N_0 \mid \frac{\partial f}{\partial x}(x) \equiv 0 \pmod p\} = \emptyset,$$

so $|N_0| = 1$ and $|S| = 0$. Since $|S| = 0$, i.e. $\frac{\partial f}{\partial x}(x) \not\equiv 0 \pmod p$, the third term of SPF will equal 0, so we get

$$\begin{aligned} Z(s) &= (p-1)p^{-1} + (1-0)p^{-1}t\left(\frac{1-p^{-1}}{1-p^{-1}t}\right) \\ &= \frac{(1-p^{-1})(1-p^{-1}t) + p^{-1}t(1-p^{-1})}{1-p^{-1}t} \\ &= \frac{(1-p^{-1})(1-p^{-1}t+p^{-1}t)}{1-p^{-1}t} \\ &= \frac{1-p^{-1}}{1-p^{-1}t}. \end{aligned}$$

Proposition 3.5. *Let f be a polynomial of the form*

1. $f(x_1, \dots, x_n) = l(x_1, \dots, x_i) + g(x_{i+1}, \dots, x_n)$ *or*
2. $f(x_1, \dots, x_n) = l(x_1, \dots, x_i) + p^e g(x_1, \dots, x_n)$

where $l(x_1, \dots, x_i) = a_1x_1 + \dots a_ix_i$, $a_j \in \mathbb{Z}$, with at least one $a_j \neq 0$ such that $p \nmid a_j$, and $e \in \mathbb{N}$. Then the Igusa local zeta function associated to f is

$$Z(s) = \frac{1-p^{-1}}{1-p^{-1}t}.$$

Proof. We will prove the proposition using SPF.
(1): Let $f(x_1, \dots, x_n) = l(x_1, \dots, x_i) + g(x_{i+1}, \dots, x_n)$, with conditions as above.

$$N_0 = \{(x_1, \dots, x_n) \in \mathbb{F}_p^n \mid f(x_1, \dots, x_n) \equiv 0 \pmod p\}.$$

We may choose any value for x_k, $k \neq j$, which then fixes x_j, so $|N_0| = p^{n-1}$. Since $l(x_1, \dots, x_i)$ is linear,

$$\frac{\partial f}{\partial x_j}(\bar{x}) = a_j,$$

which is nonzero by assumption, so SPF gives that

$$\begin{aligned} Z(s) &= (p^n - p^{n-1})p^{-n} + (p^{n-1} - 0)p^{-n}t\left(\frac{1-p^{-1}}{1-p^{-1}t}\right) \\ &= \frac{(1-p^{-1})(1-p^{-1}t) + p^{-1}t(1-p^{-1})}{1-p^{-1}t} \\ &= \frac{(1-p^{-1})(1-p^{-1}t+p^{-1}t)}{1-p^{-1}t} \\ &= \frac{1-p^{-1}}{1-p^{-1}t} \end{aligned}$$

(2): Now let $f(x_1,\ldots,x_n) = l(x_1,\ldots,x_i) + p^e g(x_1,\ldots,x_n)$, with conditions as above. Since g is multiplied by a power of p, $g \equiv 0 \pmod p$ for all values of $x_1,\ldots,x_n$, so we may choose any values for $x_{i+1},\ldots,x_n$ and only consider N_0 and S for l. Thus,

$$N_0 = \{(x_1,\ldots,x_n) \in \mathbb{F}_p^n \mid l(x_1,\ldots,x_i) \equiv 0 \pmod p\}.$$

We may choose any values for x_k, $1 \le k \le i$, $k \ne j$, which then fixes x_j, so again $|N_0| = p^{n-1}$. The set of singular points S will be the empty set since the partial derivatives of l with respect to x_j will be a_j, which is nonzero, so $|S| = 0$. Thus SPF gives the exact same formula for $Z(s)$ as for polynomials of the first form, so again

$$Z(s) = \frac{1-p^{-1}}{1-p^{-1}t}.$$

□

Example 3.6. Consider the polynomial $f(x,y) = x^3 + x + y^3$ and let $p = 2$.

$$N_0 = \{(x,y) \in \mathbb{F}_p^n \mid x^3 + x + y^3 \equiv 0 \pmod p\} = \{(0,0),(1,0)\}$$

and

$$\begin{aligned} S &= \{(x,y) \in N_0 \mid \frac{\partial f}{\partial x}(x,y) \equiv \frac{\partial f}{\partial y} \equiv 0 \pmod p\} \\ &= \{(x,y) \in N_0 \mid 3x^2 + 1 \equiv 3y^2 \equiv 0 \pmod p\} = \{(1,0)\}, \end{aligned}$$

so $|N_0| = 2 = p$ and $|S| = 1$. Thus,

$$Z(s) = (p^2 - p)p^{-2} + (p-1)p^{-2}t\left(\frac{1-p^{-1}}{1-p^{-1}t}\right) + \int_{(1+p\mathbb{Z}_p)\times p\mathbb{Z}_p} |\, x^3 + x + y^3 \,|^s \, dxdy$$

We may use the change of variables $x = 1 + px'$ and $y = py'$, which induces a change in measure $dxdy = p^{-2}dx'dy'$, so

$$Z(s) = 1-p^{-1}+p^{-2}t\left(\frac{1-p^{-1}}{1-p^{-1}t}\right)+p^{-2}\int_{\mathbb{Z}_p^2} \mid (1+px')^3+(1+px')+p^3y'^3 \mid^s dx'dy'.$$

Recall that $p=2$, so $1-p^{-1}=p^{-1}$, which gives

$$\begin{aligned} Z(s) &= p^{-1}+\frac{p^{-3}t}{1-p^{-1}t}+p^{-2}\int_{\mathbb{Z}_p^2} \mid 1+3px'+3p^2x'^2+p^3x'^3+1+px'+p^3y'^3 \mid^s dx'dy' \\ &= \frac{p^{-1}-p^{-2}t+p^{-3}t}{1-p^{-1}t}+p^{-2}\int_{\mathbb{Z}_p^2} \mid p+p^3x'+3p^2x'^2+p^3x'^3+p^3y'^3 \mid^s dx'dy'. \end{aligned}$$

We may pull a p out of the entire absolute value, giving a value of $|p|_p^s = p^{-s} = t$ in front of the integral, so

$$Z(s) = \frac{p^{-1}-p^{-2}t+p^{-3}t}{1-p^{-1}t}+p^{-2}t\int_{\mathbb{Z}_p^2} \mid 1+p^2x'+3px'^2+p^2x'^3+p^2y'^3 \mid^s dx'dy'.$$

Now the number inside the absolute value is a unit, which we know from above has absolute value 1, so

$$\begin{aligned} Z(s) &= \frac{p^{-1}-p^{-2}t+p^{-3}t}{1-p^{-1}t}+p^{-2}t \\ &= \frac{p^{-1}-p^{-2}t+p^{-3}t+p^{-2}t-p^{-3}t^2}{1-p^{-1}t} \\ &= \frac{p^{-1}+p^{-3}t-p^{-3}t^2}{1-p^{-1}t} \\ &= \frac{p^{-1}(1+p^{-2}t-p^{-2}t^2)}{1-p^{-1}t}. \end{aligned}$$

3.2. **Newton polyhedron.**

The definitions in this section can also be found in [1].

Definition 3.7. Let

$$f(x_1,\dots,x_n) = \sum_{\omega\in\mathbb{N}^n} a_\omega x_1^{\omega_1}\dots x_n^{\omega_n}$$

be a polynomial with $f(0)=0$. The *support of f*, denoted *supp*(f), is defined as

$$supp(f) = \{\omega \in \mathbb{N}^n \mid a_\omega \neq 0\}.$$

Example 3.8. Let $f(x,y) = xy - x^5 + x^2y^3$. f can be rewritten as $f(x,y) = x^1y^1 - x^5y^0 + x^2y^3$, so

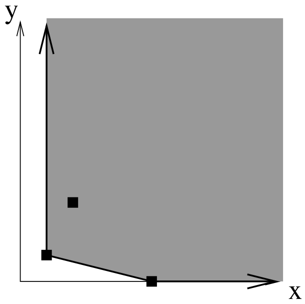
$$supp(f) = \{(1,1),(5,0),(2,3)\}.$$

Note that the coefficients of each term do not affect the point in the support, so the support point of xy, $3xy$, $-xy$ is $(1,1)$ in each of these cases.

Definition 3.9. The *Newton polyhedron* $\Gamma(f)$ of a polynomial $f(x_1,\ldots,x_n)$, $f(0,\ldots,0) = 0$ is the convex hull in $(\mathbb{R}^+)^n$ of the set

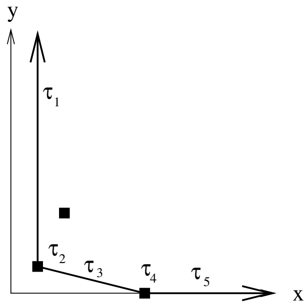
$$\bigcup_{\omega\in supp(f)} \omega + (\mathbb{R}^+)^n.$$

Example 3.10. The Newton polygon for the polynomial above, $f(x,y) = xy - x^5 + x^2y^3$, is shown below.



Definition 3.11. A *proper face* τ of a Newton polyhedron $\Gamma(f)$ is the intersection of $\Gamma(f)$ with a supporting hyperplane that does not intersect the interior of $\Gamma(f)$. The improper face of $\Gamma(f)$ is itself $\Gamma(f)$. A *facet* of $\Gamma(f)$ is a face with dimension equal to $n-1$, where n is the dimension of $\Gamma(f)$.

Example 3.12. The figure below gives the same Newton polygon as above, with the faces labelled. In this example, the facets of $\Gamma(f)$ are τ_1, τ_3, and τ_5.



$$\begin{aligned}
\tau_1 &= \{(1,y) \mid y > 1\}\\
\tau_2 &= \{(1,1)\}\\
\tau_3 &= \{t(1,1) + (1-t)(5,0) \mid 0 \le t \le 1\}\\
\tau_4 &= \{(5,0)\}\\
\tau_5 &= \{(x,0) \mid x > 5\}
\end{aligned}$$

Definition 3.13. For $a \in (\mathbb{R}^+)^n$, define

$$m(a) := \min_{x\in supp(f)} \{a\cdot x\}$$

and define the *first meet locus of a* as

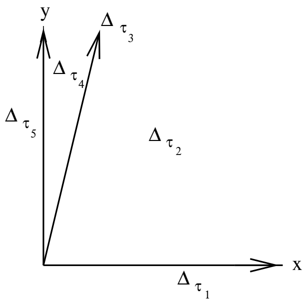
$$F(a) := \{x \in \Gamma(f) \mid a\cdot x = m(a)\}.$$

Definition 3.14. The *cone associated to a face* τ *of* $\Gamma(f)$, Δ_τ, is defined as

$$\Delta_\tau = \{a \in (\mathbb{R}^+)^n \mid F(a) = \tau\}.$$

Example 3.15. The table and figure below give the supporting hyperplanes of the facets τ and the spanning vectors of the cones Δ_τ associated to each face τ of the Newton polygon shown above.

$\boldsymbol{i}$	$\boldsymbol{\tau_i}$ (or supporting hyperplane for $\boldsymbol{\tau_i}$ a facet)	spanning vectors for $\boldsymbol{\Delta_{\tau_i}}$
1	$y > 0$	$\{(1,0)\}$
2	$(1,1)$	$\{(1,0),(1,4)\}$
3	$x + 4y = 5$	$\{(1,4)\}$
4	$(5,0)$	$\{(1,4),(0,1)\}$
5	$x > 0$	$\{(0,1)\}$



Definition 3.16. The polynomial $f_\tau(x_1, \ldots, x_n)$ is defined as

$$f\tau = \sum_{\omega \in supp(f) \cap \tau} a_\omega x_1^{\omega_1} \ldots x_n^{\omega_n}.$$

Definition 3.17. A polynomial $f(x_1, \ldots, x_n)$ is *nondegenerate with respect to all faces of its Newton polyhedron* if the system of equations

$$\begin{cases} f_\tau(x_1, \ldots, x_n) = f_\tau(\bar{x}) \equiv 0 \pmod{p} \\ \frac{\partial f_\tau}{\partial x_i}(\bar{x}) \equiv 0 \pmod{p} \end{cases}$$

has no solutions in $(\mathbb{F}_p^*)^n$ for all faces τ of the Newton polyhedron.

Kathleen Hoornaert introduced a formula for the Igusa local zeta function associated with a polynomial in n variables that is nondegenerate with respect to all faces of its Newton polyhedron $\Gamma(f)$. [1]

Theorem 3.18. *Let* $f(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$ *be a polynomial with* $f(0, \ldots, 0) = 0$ *and let* f *be non-degenerate with respect to all faces of its Newton polyhedron. Then*

$$Z(s) = \sum_{faces\ \tau \in \Gamma(f)} L_\tau S_{\Delta_\tau},$$

where the sum is over the faces τ of the Newton polyhedron, with

$$L_\tau = p^{-n}\left((p-1)^n - p|N_\tau|\left(\frac{p^s-1}{p^{s+1}-1}\right)\right),$$

$$N_\tau = \{(x_1,\dots,x_n) \in (\mathbb{F}_p^*)^n \mid f_\tau(x_1,\dots,x_n) \equiv 0 \pmod{p}\},$$

$$S_{\Delta_\tau} = \sum_{\mathbf{k}} p^{-(\sigma(\mathbf{k})+m(\mathbf{k})s)},\ \mathbf{k} \in \Delta_\tau \cap \mathbb{N}^n,$$

$$\sigma(\mathbf{k}) = \sum_{i=1}^{n} k_i,$$

and

$$m(\mathbf{k}) = \min_{b\in supp(f)}\{\mathbf{k}\cdot\mathbf{b}\}.$$

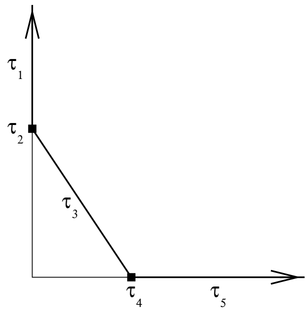
Example 3.19. Consider the polynomial in two variables

$$f(x,y) = x^2 + y^3.$$

For this polynomial, we have that

$$supp(f) = \{(2,0),(0,3)\}.$$

The Newton polygon is given by the figure below.



The table below lists the supporting hyperplane to each τ for this polynomial, giving the spanning vectors of the cone Δ_τ associated with it, f_τ, L_τ, and S_τ.

Information for $f(x,y) = x^2 + y^3$					
$\boldsymbol{\tau_i}$	$\boldsymbol{\tau}$ (hyperplane)	$\boldsymbol{\Delta_\tau}$ (spanning vectors)	$\boldsymbol{f_\tau}$	$\boldsymbol{L_\tau}$	$\boldsymbol{S_{\Delta_\tau}}$
τ_1	$y \geq 3$	$\{(1,0)\}$	$f_{\tau_1} = y^3$	$p^{-2}(p-1)^2$	$\frac{1}{p-1}$
τ_2	$(0,3)$	$\{(1,0),(3,2)\}$	$f_{\tau_2} = y^3$	$p^{-2}(p-1)^2$	$\frac{(p^3+t^3)t^3}{(p-1)(p^5-t^6)}$
τ_3	$3x+2y = 3$	$\{(3,2)\}$	$f_{\tau_3} = x^2 + y^3$	$p^{-2}((p-1)^2 - \frac{(p-1)(1-t)}{p-t})$	$\frac{t^6}{p^5-t^6}$
τ_4	$(2,0)$	$\{(3,2),(0,1)\}$	$f_{\tau_4} = x^2$	$p^{-2}(p-1)^2$	$\frac{(p^4+p^2t^2+t^4)t^2}{(p-1)(p^5-t^6)}$
τ_5	$x \geq 2$	$\{(0,1)\}$	$f_{\tau_5} = x^2$	$p^{-2}(p-1)^2$	$\frac{1}{p-1}$
τ_6	$\Gamma(f)$	$\{(0,0)\}$	$f_{\tau_6} = x^2 + y^3$	$p^{-2}(p-1)^2$	1

Given this information, to compute the zeta function for this polynomial, using the convention $t = p^{-s}$, we get

$$Z(s) = \frac{(p-1)(p^5 - p^3t + p^3t^2 - t^5)}{(p-t)(p^5 - t^6)}.$$

4. Degenerate Polynomials

The formula outlined above is only correct for polynomials that are nondegenerate with respect to all faces of their Newton polyhedra. The formula fails when computing L_τ for faces for which f_τ is degenerate. The formula stated above, $L_\tau = p^{-n}\left((p-1)^n - p|N_\tau|\left(\frac{p^s-1}{p^{s+1}-1}\right)\right)$ assumes that there are no singular points in $(\mathbb{F}_p^*)^n$, i.e. no solutions for which any coordinate is zero (see the definition of 'nondegeneracy' stated above). If f is degenerate and there ARE singular points in $(\mathbb{F}_p^*)^n$, the formula must be modified to take these points into consideration. Thus, for f_τ degenerate, let

$$\overline{L_\tau} = p^{-n}((p-1)^n - |N_\tau|) + (|N_\tau| - |S|)p^{-n}t\left(\frac{1-p^{-1}}{1-p^{-1}t}\right).$$

Then the formula for the zeta function is

$$Z(s) = \sum_{\tau\,\text{nondeg.}} L_\tau S_{\Delta_\tau}$$

$$+\sum_{\tau\,\text{deg.}}\left(\overline{L_\tau} S_{\Delta\tau} + \sum_{\mathbf{k}\in\mathbb{N}\cap\Delta_\tau} p^{-(\sigma(\mathbf{k})+m(\mathbf{k})s)}\sum_{\alpha\in S}\int_{\alpha+p\mathbb{Z}_p^n} |\ f_\tau + p\tilde{f}\ |^s\, du_1\ldots du_n\right),$$

where $p^{-m(\mathbf{k})}f = f_\tau + p\tilde{f}$.

Proof. Proof of formula:
From [1],

$$Z(s) = \sum_\tau \sum_{\mathbf{k}\in\Delta_\tau\cap\mathbb{N}^n}\int_{p^{k_1}u_1\times p^{k_2}u_2\times\ldots\times p^{k_n}u_n} |\ f(x_1,\ldots,x_n)\ |^s\, dx_1,\ldots,dx_n.$$

We already know the formula holds for the nondegenerate faces. For the degenerate faces, we want to compute

$$I_\tau = \sum_{\mathbf{k}\in\Delta_\tau\cap\mathbb{N}^n}\int_{p^{k_1}u_1\times p^{k_2}u_2\times\ldots\times p^{k_n}u_n} |\ f(x_1,\ldots,x_n)\ |^s\, dx_1\ldots dx_n.$$

First we will change the variables, using $x_i = p^{k_i}u_i$, $dx = p^{-k_i}du_i$, so we get

$$I_\tau = \sum_{\mathbf{k}\in\Delta_\tau\cap\mathbb{N}^n} p^{-\sigma(\mathbf{k})}\int_{(\mathbb{Z}_p^*)^n} |\ f(p^{k_1}u_1,\ldots,p^{k_n}u_n)\ |^s\, du_1\ldots du_n.$$

By definition, the maximum number of p's that we will be able to pull out from f will be $m(\mathbf{k})$, so we get

$$I_\tau = \sum_{\mathbf{k}} p^{-(\sigma(\mathbf{k})+m(\mathbf{k})s)} \int_{(\mathbb{Z}_p^*)^n} \mid f_\tau + p\tilde{f} \mid^s du_1 \ldots du_n.$$

We can now apply SPF to the integral, which gives us

$$I_\tau = \sum_{\mathbf{k}} p^{-(\sigma(\mathbf{k})+m(\mathbf{k})s)} \left(p^{-n}((p-1)^n - |N_\tau|) + (|N_\tau| - |S_\tau|) p^{-n} t \left(\frac{1-p^{-1}}{1-p^{-1}t} \right) \right)$$

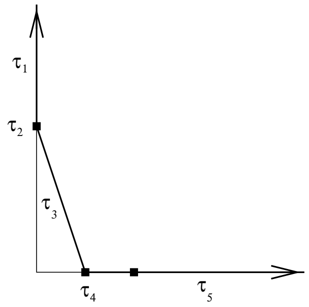
$$+ \sum_{\mathbf{k}} p^{-(\sigma(\mathbf{k})+m(\mathbf{k})s)} \left(\sum_{\alpha \in S} \int_{\alpha + (p\mathbb{Z}_p^*)^n} \mid f_\tau + \tilde{f} \mid^s du \right).$$

Using our notation above, I_τ can be rewritten as

$$I_\tau = \overline{L}_\tau S_{\Delta_\tau} + \sum_{\mathbf{k}} p^{-(\sigma(\mathbf{k})+m(\mathbf{k})s)} \sum_{\alpha \in S} \int_{\alpha + (p\mathbb{Z}_p^*)^n} \mid f_\tau + p\tilde{f} \mid^s du_1 \ldots du_n$$

for τ a degenerate face of the Newton polyhedron. □

Example 4.1. Consider the polynomial $f(x,y) = x^3 + x + y^3$. Its Newton polygon is shown below.



The polynomial is degenerate with respect to $\Gamma(f)$ only for $p = 2$, so I will compute it in that case. For the nondegenerate faces, we can simply apply the original formula, which gives the following table.

Information for $f(x,y) = x^3 + x + y^3$					
$\boldsymbol{i}$	$\boldsymbol{\tau}$ (hyperplane)	$\boldsymbol{\Delta_{\tau_i}}$ (spanning vectors)	$\boldsymbol{f_{\tau_i}}$	$\boldsymbol{L_\tau}$	$\boldsymbol{S_{\Delta_\tau}}$
1	$y > 0$	$\{(1,0)\}$	y^3	$p^{-2}(p-1)^2$	$\frac{1}{p-1}$
2	$(0,3)$	$\{(1,0),(3,1)\}$	y^3	$p^{-2}(p-1)^2$	$\frac{1}{p-1}$
3	$3x + y = 3$	$\{(3,1)\}$	$x + y^3$	$p^{-2}(p-1)^2 - \frac{(p^{-1}(p-1)(p^s-1)}{p^{s+1}-1}$	$\frac{1}{p-1}$
4	$(1,0)$	$\{(3,1),(0,1)\}$	x	$p^{-2}(p-1)^2$	$\frac{1+p^2t^{-1}+p^3t^{-2}}{(p^4t^{-3}-1)(p-1)}$
5	$x > 0$	$\{(0,1)\}$	$x + x^3$	degenerate	1
6	$\Gamma(f)$	$\{(0,0)\}$	$x^3 + x + y^3$	$p^{-2}(p-1)^2$	1

The only I_τ which we have not computed is the one associated with τ_5, for which $f_{\tau_5} = x + x^3$ is degenerate. (Note that the point $(1,1)$ satisfies both this equation and its derivatives modulo 2.) We will calculate I_{τ_5} both from the definiton and from the formula above.

From the definition:
We want to calculate

$$I_{\tau_5} = \sum_{\mathbf{k}} \int_{p^{k_1}u_1 \times p^{k_2}u_2} \mid x^3 + x + y^3 \mid^s dxdy,$$

where $\Delta_{\tau_5} = \{\lambda(0,1) \mid \lambda > 0\}$, so $k = (0,\lambda)$ for $\lambda > 0$, $\lambda \in \mathbb{N}$.

$$\begin{aligned}
I_{\tau_5} &= \sum_{\lambda=1}^{\infty} \int_{u_1 \times p^\lambda u_2} \mid x^3 + x + y^3 \mid^s dxdy \\
&= \sum_{\lambda=1}^{\infty} p^{-\lambda} \int_{(\mathbb{Z}_p \setminus p\mathbb{Z}_p)^2} \mid u_1^3 + u_1 + p^{3\lambda}u_2^3 \mid^s du_1du_2 \\
&= \sum_{\lambda=1}^{\infty} p^{-\lambda}p^{-2} \int_{\mathbb{Z}_p^2} \mid (1+px')^3 + (1+px') + p^{3\lambda}(1+py')^3 \mid^s dx'dy' \\
&= \sum_{\lambda=1}^{\infty} p^{-\lambda}p^{-2} \int_{\mathbb{Z}_p^2} \mid 1 + 3px' + 3p^2x'^2 + p^3x'^3 + 1 + px' + p^{3\lambda}(1+py')^3 \mid^s dx'dy' \\
&= \sum_{\lambda=1}^{\infty} p^{-\lambda}p^{-2} \int_{\mathbb{Z}_p^2} \mid p + p^3x' + 3p^2x'^2 + p^3x'^3 + p^{3\lambda}(1+py')^3 \mid^s dx'dy' \\
&= \sum_{\lambda=1}^{\infty} p^{-\lambda}p^{-2}t \int_{\mathbb{Z}_p^2} \mid 1 + p^2(px' + 3x'^2 + px'^3 + p^{3\lambda-2}(1+py')^3) \mid^s dx'dy'
\end{aligned}$$

$$\begin{aligned} I_{\tau_5} &= \sum_{\lambda=1}^{\infty} p^{-\lambda} p^{-2} t \\ &= p^{-2} t \left(\frac{p^{-1}}{1-p^{-1}} \right) \\ &= p^{-2} t. \end{aligned}$$

From the formula above:

$$I_{\tau_5} = \overline{L}_{\tau_5} S_{\Delta_{\tau_5}} + \sum_{\mathbf{k}} p^{-(\sigma(\mathbf{k})+m(\mathbf{k})s)} \int_{s+(p\mathbb{Z}_p^*)^2} \mid f_\tau + p\tilde{f} \mid^s du_1 du_2.$$

We know that $\mathbf{k} = (0, \lambda)$, so $\sigma(\mathbf{k}) = \lambda$ and

$$m(\mathbf{k}) = \min_{\mathbf{b} \in supp(f)} \{\mathbf{k} \cdot \mathbf{b}\} = (0, \lambda) \cdot (1, 0) = 0.$$

To find $\overline{L_{\tau_5}}$, we must first find

$$\begin{aligned} |N_{\tau_5}| &= |\{(x,y) \in (\mathbb{F}_p^*)^n \mid x^3 + x + y^3 \equiv 0 \pmod 2\}| = |\{(0,0),(1,0)\}| \\ &= 2 = p \end{aligned}$$

and

$$|S_{\tau_5}| = |\{(x,y) \in N_{\tau_5} \mid f_x \equiv f_y \equiv 0 \pmod 2\}| = |\{(1,0)\}| = 1.$$

Thus,

$$\begin{aligned} \overline{L_{\tau_5}} &= (p^2 - p)p^{-2} + (p-1)p^{-2} t \left(\frac{1-p^{-1}}{1-p^{-1}t} \right) \\ &= 1 - p^{-1} + p^{-2} t \left(\frac{p^{-1}}{1-p^{-1}t} \right) \\ &= \frac{p^{-1}(1-p^{-1}t) + p^{-3}t}{1-p^{-1}t} \\ &= \frac{p^{-1} - p^{-2}t + p^{-3}t}{1-p^{-1}t}. \end{aligned}$$

Finally, we need to calculate

$$\begin{aligned}
I &= \sum_{\mathbf{k}} p^{-(\sigma(\mathbf{k})+m(\mathbf{k})s)} \sum_{\alpha\in S} \int_{\alpha+(p\mathbb{Z}_p^*)^2} \mid f_\tau + p\tilde{f} \mid^s du_1du_2 \\
&= \sum_{\lambda=1}^{\infty} p^{-\lambda} \int_{(1+p\mathbb{Z}_p)\times p\mathbb{Z}_p} \mid u_1^3 + u_1 + p^{3\lambda}u_2 \mid^s du_1du_2 \\
&= \sum_{\lambda=1}^{\infty} p^{-\lambda}p^{-2} \int_{\mathbb{Z}_p^2} \mid 1 + 3px + 3p^2x^2 + p^3x^3 + 1 + px + p^{3\lambda}p^3y^3 \mid^s dxdy \\
&= \sum_{\lambda=1}^{\infty} p^{-\lambda}p^{-2} \int_{\mathbb{Z}_p^2} \mid p + p^3x + 3p^2x^2 + p^3x^3 + p^{3\lambda}p^3y^3 \mid^s dxdy \\
&= \sum_{\lambda=1}^{\infty} p^{-\lambda}p^{-2}t \int_{\mathbb{Z}_p^2} \mid 1 + p^2(px + 3x^2 + px^3 + p^{3\lambda+1}y^3 \mid^s dxdy \\
&= \sum_{\lambda=1}^{\infty} p^{-\lambda}p^{-2}t \\
&= p^{-2}t\left(\frac{p^{-1}}{1-p^{-1}}\right) \\
&= p^{-2}t.
\end{aligned}$$

By summing the components from all the faces, we get that

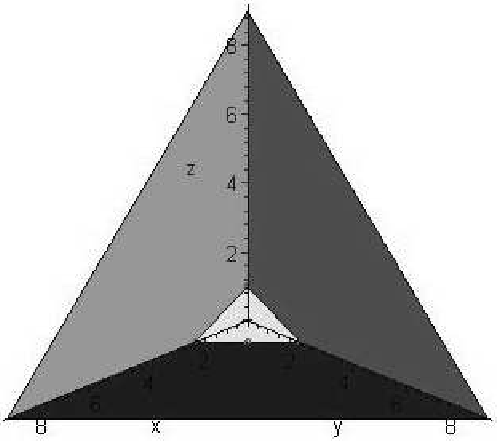
$$Z(s) = \frac{p^{-1} + p^{-3}t - p^{-3}t^2}{1 - p^{-1}t}$$

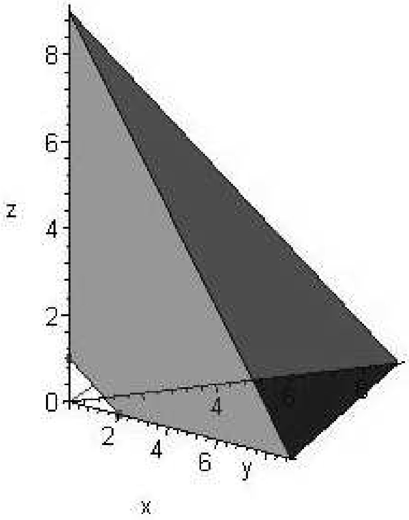
where we are assuming here that $p = 2$.

Example 4.2. The polynomial $f(x, y, z) = (x - y)^2 + z$ is degenerate with respect to two faces of its Newton polyhedron. Because f can be written as $f(x, y, z) = g(x, y) + z$, where $g(x, y) = (x - y)^2$, from **Proposition 3.5** we know that the Igusa local zeta function associated to f is

$$Z(s) = \frac{1 - p^{-1}}{1 - p^{-1}t}.$$

Two different views of its Newton polyhedron are shown below. These images were created using the computer program *Polygusa* [5] and cdd. [2]





The polyhedron has 4 facets and 14 faces. The supporting hyperplanes, along with the spanning vectors of the cones associated to each face, f_τ, L_τ, and S_τ are listed in the table below.

$\boldsymbol{i}$	$\boldsymbol{\tau_i}$ (hyperplane)	$\boldsymbol{\Delta_{\tau_i}}$ (spanning vectors)	$\boldsymbol{f_{\tau_i}}$	$\boldsymbol{L_{\tau_i}}$	$\boldsymbol{S_{\Delta_{\tau_i}}}$
1	$y > 0, z > 0$	$(1,0,0)$	$y^2 + z$	$p^{-3}(p-1)^3 - \frac{p^{-3}(1-t)}{1-p^{-1}t}$	$\frac{p^{-1}}{1-p^{-1}}$
2	$x > 0, z > 0$	$(0,1,0)$	$x^2 + z$	$p^{-3}(p-1)^3 - \frac{p^{-3}(1-t)}{1-p^{-1}t}$	$\frac{p^{-1}}{1-p^{-1}}$
3	$x > 0, y > 0$	$(0,0,1)$	$(x-y)^2$	degenerate	$\frac{p^{-1}}{1-p^{-1}}$
4	$x + y + 2z \geq 2$	$(1,1,2)$	$(x-y)^2 + z$	$p^{-3}(p-1)^3 - \frac{p^{-3}(p-1)^2(1-t)}{1-p^{-1}t}$	$\frac{p^{-4}t^2}{1-p^{-4}t^2}$
5	$y > 0$	$(1,0,0),(0,0,1)$	y^2	$p^{-3}(p-1)^3$	$\frac{2p^{-1}}{1-p^{-1}}$
6	$z > 0$	$(1,0,0),(0,1,0)$	z	$p^{-3}(p-1)^3$	$\frac{2p^{-1}}{1-p^{-1}}$
7	$x > 0$	$(0,1,0),(0,0,1)$	x^2	$p^{-3}(p-1)^3$	$\frac{2p^{-1}}{1-p^{-1}}$
8	$x + y = 1$	$(0,0,1),(1,1,2)$	$(x-y)^2$	degenerate	$\frac{p^{-1}(1+p^{-3}t^2-2p^{-4}t^2)}{(1-p^{-1})(1-p^{-4}t^2)}$
9	$y + 2z = 2$	$(1,0,0),(1,1,2)$	$y^2 + z$	$p^{-3}(p-1)^3 - \frac{p^{-3}(p-1)^2(1-t)}{1-p^{-1}t}$	$\frac{p^{-1}(1+p^{-3}t^2-2p^{-4}t^2)}{(1-p^{-1})(1-p^{-4}t^2)}$
10	$x + 2z = 2$	$(0,1,0),(1,1,2)$	$x^2 + z$	$p^{-3}(p-1)^3 - \frac{p^{-3}(p-1)^2(1-t)}{1-p^{-1}t}$	$\frac{p^{-1}(1+p^{-3}t^2-2p^{-4}t^2)}{(1-p^{-1})(1-p^{-4}t^2)}$
11	$(0,2,0)$	$(1,0,0),(0,0,1),(1,1,2)$	y^2	$p^{-3}(p-1)^3$	$\frac{p^{-1}(2+p^{-3}t^2-3p^{-4}t^2)}{(1-p^{-1})(1-p^{-4}t^2)}$
12	$(0,0,1)$	$(1,0,0),(0,1,0),(1,1,2)$	z	$p^{-3}(p-1)^3$	$\frac{p^{-1}(2+p^{-3}t^2-3p^{-4}t^2)}{(1-p^{-1})(1-p^{-4}t^2)}$
13	$(2,0,0)$	$(0,1,0),(0,0,1),(1,1,2)$	x^2	$p^{-3}(p-1)^3$	$\frac{p^{-1}(2+p^{-3}t^2-3p^{-4}t^2)}{(1-p^{-1})(1-p^{-4}t^2)}$
14	$\Gamma(f)$	$(0,0,0)$	$(x-y)^2 + z$	$p^{-3}(p-1)^3 - \frac{p^{-3}(p-1)^2(1-t)}{1-p^{-1}t}$	1

Using the formula given earlier, we can compute I_τ for the two degenerate faces of f.

τ_8:
The cone $\Delta_{\tau_8} = \{\lambda_1(0,0,1) + \lambda_2(1,1,2) \mid \lambda > 0\}$, so $\mathbf{k} = (\lambda_2, \lambda_2, \lambda_1 + 2\lambda_2)$, $\sigma(\mathbf{k}) = \lambda_1 + 4\lambda_2$, and $m(\mathbf{k}) = 2\lambda_2$. Thus,

$$\begin{aligned} &f_\tau(u_1,u_2,u_3) + p\tilde{f}(u_1,u_2,u_3) \\ =&(u_1-u_2)^2 + p^{\lambda_1}u_3. \end{aligned}$$

$|N_\tau| = |S_\tau| = (p-1)^2$, since we may choose any value for u_1 and u_3 between 1 and $p-1$, while u_2 must equal u_1. Note that $S_\tau = \{(u_1,u_1,u_3)\}$.

$$\overline{L_{\tau_8}} = ((p-1)^3 - (p-1)^2)p^{-3}.$$

The last thing to compute is the sum over the cone of the integral of the singular points:

$$
\begin{aligned}
I &= \sum_{\lambda_1}\sum_{\lambda_2} p^{-\lambda_1-4\lambda_2} \sum_{(u_1,u_1,u_3)} \int_{(u_1,u_1,u_3)+(p\mathbb{Z}_p^*)^n} \mid (u_1-u_2)^s + p^{\lambda_1}u_3 \mid^s du_1du_2du_3 \\
&= \sum_{\lambda_1}\sum_{\lambda_2} p^{-\lambda_1-4\lambda_2}(p-1) \sum_{(u_1,u_2,u_3)} \int_{(u_1,u_1,u_3)+(p\mathbb{Z}_p^*)^n} \mid p^{\lambda_1}u_3 \mid^s du_1du_2du_3 \\
&= \sum_{\lambda_1}\sum_{\lambda_2} p^{-\lambda_1-4\lambda_2} t^{\lambda_1}(p-1)(1-p^{-1})\left(\frac{1-p^{-1}}{1-p^{-1}t}\right).
\end{aligned}
$$

So

$$
I_{\tau_8} = ((p-1)^3-(p-1)^2)p^{-3}\left(\frac{p^{-5}t^2}{(1-p^{-1})(1-p^{-4}t^2)}\right)
$$

$$
+(p-1)\left(\frac{(1-p^{-1})^3}{1-p^{-1}t}\right)\left(\frac{p^{-5}t^2}{(1-p^{-1})(1-p^{-4}t^2)}\right).
$$

We may follow the same procedure for τ_3, the other degenerate face, which gives

$$
I_{\tau_3} = ((p-1)^3-(p-1)^2)p^{-3}\left(\frac{p^{-1}}{1-p^{-1}}\right) + (p-1)\left(\frac{(1-p^{-1})^3}{1-p^{-1}t}\right)\left(\frac{p^{-1}t}{1-p^{-1}t}\right).
$$

By summing I_τ for all τ, we get

$$
Z(s) = \frac{1-p^{-1}}{1-p^{-1}t}.
$$

5. CONCLUSION

For further research, it would be interesting to study classes of polynomials for which we can determine more about the singular integral in the Newton polyhedron method for the faces for which the polynomial is degenerate. Because the integral may depend on the cone, we have not been able to generalize it further, but it may be possible for certain classes of polynomials. It would also be interesting to consider classes of polynomials, both degenerate and non-degenerate, that have the same associated Igusa local zeta function and to study what makes some polynomials degenerate as opposed to others. An example of a class of these polynomials would be the polynomials with linear terms described in this paper, which all have the same zeta function, regardless of whether or not the polynomial is degenerate with respect to its Newton polyhedron.

Acknowledgements

I would like to thank Dr. Margaret Robinson for all her help this summer, as well as Dr. Jessica Sidman, all the members of my REU group (Rex Cheung, Rosi Dineva, Joanna Miles, Ricardo Portilla, Matthew Pragel, and the Polytope group) and Joel Louwsma for his help in proofreading this paper. I would also like to acknowledge Kathleen Hoornaert and Davy Loots for the computer program *Polygusa* and Joanna Miles for the program *Zeros*, which were an immense help in performing calculations relating to the Igusa local zeta function. Finally, I would like to thank the NSF for providing the funding for this work.

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