Igusa local zeta functions and p-adic analysis

Newton polyhedra and degenerate polynomials

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Introduction

An introduction to p-adic valuation...
Given a number $a \in \mathbb{Q}$, the *p-adic absolute value of a*, denoted $|a|_p$, is defined as

$$|a|_p = \begin{cases} p^{-ord_p(a)} & \text{if } a \neq 0 \\ 0 & \text{if } a = 0. \end{cases}$$

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- $ord_5(\frac{1}{25}) = -2 \Rightarrow |\frac{1}{25}|_5 = 5^2 = 25$
- $ord_3(18) = 2 \Rightarrow |18|_3 = 3^{-2} = \frac{1}{9}$

p-adic numbers

- The field of all p-adic numbers, $\mathbb{Q}_p$, is defined as all equivalence classes of p-adic Cauchy sequences. A sequence $\{x_i\}$ is Cauchy if for all ϵ there exists $N \in \mathbb{N}$ such that if $m, n > N$, then $|x_n - x_m| < \epsilon$.

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- The ring of p-adic integers, $\mathbb{Z}_p$, is composed of all p-adic numbers $a \in \mathbb{Q}_p$ with $|a|_p \leq 1$.

Every p-adic integer a has the form

$$a = a_0 + pa_1 + p^2 a_2 + \ldots + p^m a_m + \ldots$$

for some $m \in \mathbb{Z}$.

p-adic numbers

The *units* in $\mathbb{Z}_p$ are all p-adic integers a with

$$|a|_p = 1.$$

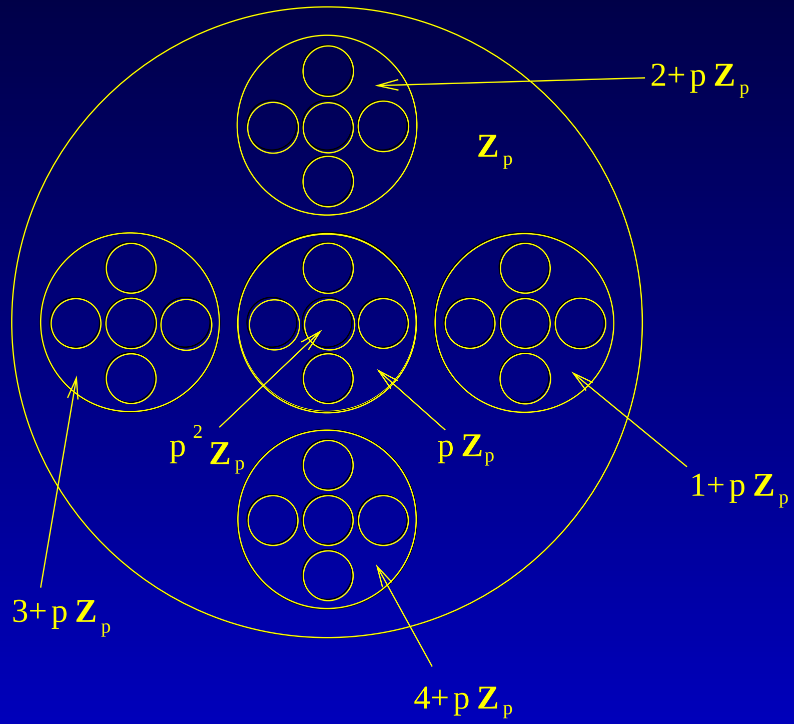
i.e. p-adic integers of the form

$$a = a_0 + pa_1 + \ldots + p^m a_m + \ldots$$

with $a_0 \neq 0$.

Topology

The topology of $\mathbb{Z}_p$, for $p = 5$:



Note that every point in an open ball is a center of that ball.

Igusa local zeta function

The Igusa local zeta function associated to a polynomial $f(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$ is defined as

$$Z(s) = \int_{\mathbb{Z}_p^n} \mid f(x_1, \ldots, x_n) \mid_p^s dx_1 \ldots dx_n,$$

$s \in \mathbb{C}$, $Re(s) > 0$.

We use the convention $t = p^{-s}$.

Stationary Phase Formula

$$Z(s) = (p^n - |N_0|)p^{-n} + (|N_0| - |S|)p^{-n}t\left(\frac{1-p^{-1}}{1-p^{-1}t}\right)$$

$$+ \sum_{\alpha \in S} \int_{\alpha + p\mathbb{Z}_p^n} |\ f(x_1, \ldots, x_n)\ |^s \, dx_1 \ldots dx_n$$

where

$$N_0 = \{(x_1, \ldots, x_n) \in \mathbb{F}_p^n \mid f(x_1, \ldots, x_n) \equiv 0 \pmod{p}\}$$

and

$$S = \{(x_1, \ldots, x_n) \in N_0 \mid \frac{\partial f}{\partial x_i}(x) \equiv 0 \pmod{p}, 1 \leq i \leq n\}.$$

Ex. 1 - $f(x) = x$

$$N_0 = \{x \mid x \equiv 0 \pmod p\} \Rightarrow |N_0| = 1$$

$$S = \{x \in N_0 \mid \frac{\partial f}{\partial x}(x) \equiv 0 \pmod p\} = \emptyset \Rightarrow |S| = 0,$$

so using SPF...

$$Z(s) = (p-1)p^{-1} + (1-0)p^{-1}t\left(\frac{1-p^{-1}}{1-p^{-1}t}\right)$$

$$= \frac{1-p^{-1}}{1-p^{-1}t}$$

Ex. 2 - $f(x,y,z) = (x-y)^2 + z$

$$N_0 = \{(x, y, -(x-y)^2)\} \Rightarrow |N_0| = p^2$$

$$S = \emptyset \Rightarrow |S| = 0$$

Zeta function:

$$Z(s) = (p^3 - p^2)p^{-3} + (p^2 - 0)p^{-3}t\left(\frac{1-p^{-1}}{1-p^{-1}t}\right)$$

$$= \frac{1-p^{-1}}{1-p^{-1}t}$$

Support of f

Given a polynomial

$$f(x_1, \ldots, x_n) = \sum_{\omega \in \mathbb{N}^n} a_\omega x_1^{\omega_1} \ldots x_n^{\omega_n},$$

the support of f is defined as

$$supp(f) = \{\omega \in \mathbb{N}^n \mid a_\omega \neq 0\}.$$

Ex)

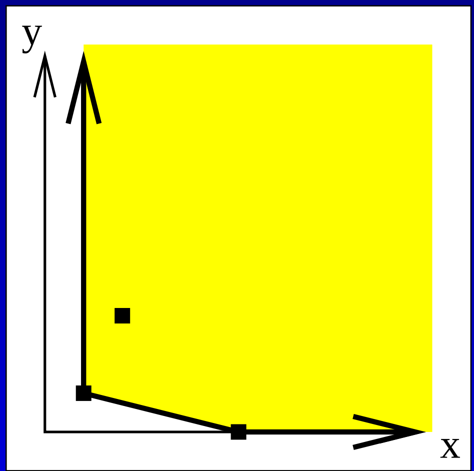
$$f(x, y) = xy - x^5 + x^2 y^3$$

$$supp(f) = \{(1,1), (5,0), (2,3)\}$$

Newton polyhedron

The *Newton polyhedron* $\Gamma(f)$ of a polynomial $f(x_1, \ldots, x_n)$, f(0)=0, is the convex hull in $(\mathbb{R}^+)^n$ of the set

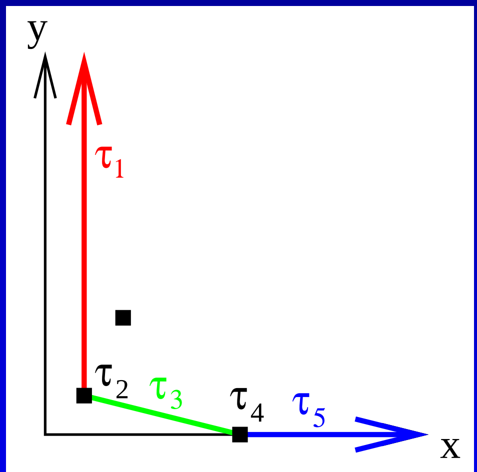
$$\bigcup_{\omega \in supp(f)} \omega + (\mathbb{R}^+)^n.$$



$$f(x,y) = xy - x^5 + x^2y^3$$

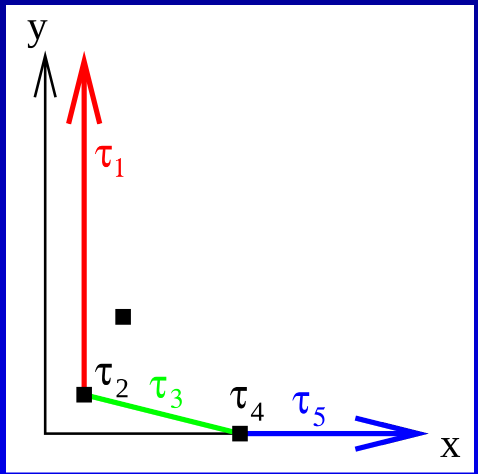
Faces and associated cones

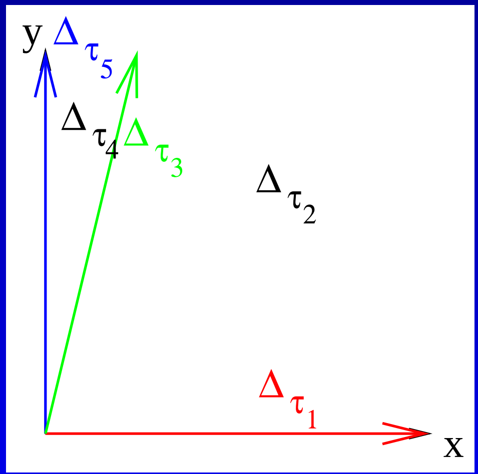
- A *face* τ of $\Gamma(f)$ is the intersection of $\Gamma(f)$ with a supporting hyperplane that does not intersect the interior of $\Gamma(f)$. A *facet* is a face of dimension $n-1$.



Faces and associated cones

- A *face* τ of $\Gamma(f)$ is the intersection of $\Gamma(f)$ with a supporting hyperplane that does not intersect the interior of $\Gamma(f)$. A *facet* is a face of dimension $n-1$.
- The *cone* associated to a facet τ is the normal vector to τ. The cone for a face that is not a facet is the span of the cones for all facets containing the face.





Degeneracy

Given a polynomial $f(x_1, \ldots, x_n)$, the polynomial f_τ is composed of the terms of f whose support is equal to $supp(f) \cap \tau$.

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A polynomial $f(x_1, \ldots, x_n)$ is *non-degenerate* with respect to all faces of its Newton polyhedron if the system

$$\begin{cases} f_\tau(x_1, \ldots, x_n) \equiv 0 \pmod p \\ \frac{\partial f_\tau}{\partial x_i}(x) \equiv 0 \pmod p \end{cases}$$

has no non-zero solutions.

$\sigma(\mathbf{k})$ and $m(\mathbf{k})$

For an n-vector $\mathbf{k}$,

$$\sigma(\mathbf{k}) := \sum_{i=1}^{n} k_i$$

and

$$m(\mathbf{k}) := inf_{x\in\Gamma(f)}\{\mathbf{k}\cdot\mathbf{x}\}.$$

Non-degenerate polynomials

For a polynomial $f(x_1, \ldots, x_n)$ that is nondegenerate with respect to all faces of its Newton polyhedron, the Igusa local zeta function associated to f is

$$Z(s) = \sum_{\tau \in \Gamma(f)} L_\tau S_{\Delta_\tau},$$

where

$$L_\tau = p^{-n} \left((p-1)^n - p|N_\tau| \left(\frac{p^s - 1}{p^{s+1} - 1} \right) \right),$$

$$N_\tau = \{(x_1, \ldots, x_n) \in (\mathbb{F}_p^*)^n \mid f_\tau(x_1, \ldots, x_n) \equiv 0 \pmod{p}\},$$

$$S_{\Delta_\tau} = \sum_{\mathbf{k}} p^{-(\sigma(\mathbf{k}) + m(\mathbf{k})s)}.$$

Degenerate polynomials

For a polynomial which is denegerate with respect to some faces of its Newton polyhedron, S_{Δ_τ} doesn't change, but L_τ does.

$$\overline{L_\tau} = p^{-n}((p-1)^n - |N_\tau|) + (|N_\tau| - |S|)p^{-n} t \left(\frac{1 - p^{-1}}{1 - p^{-1}t} \right).$$

Note that for all faces for which f_τ is non-degenerate, L_τ remains as in the original formula.

Degenerate polynomials 2

$$Z(s) = \sum_{\tau \text{ nondeg.}} L_\tau S_{\Delta_\tau}$$

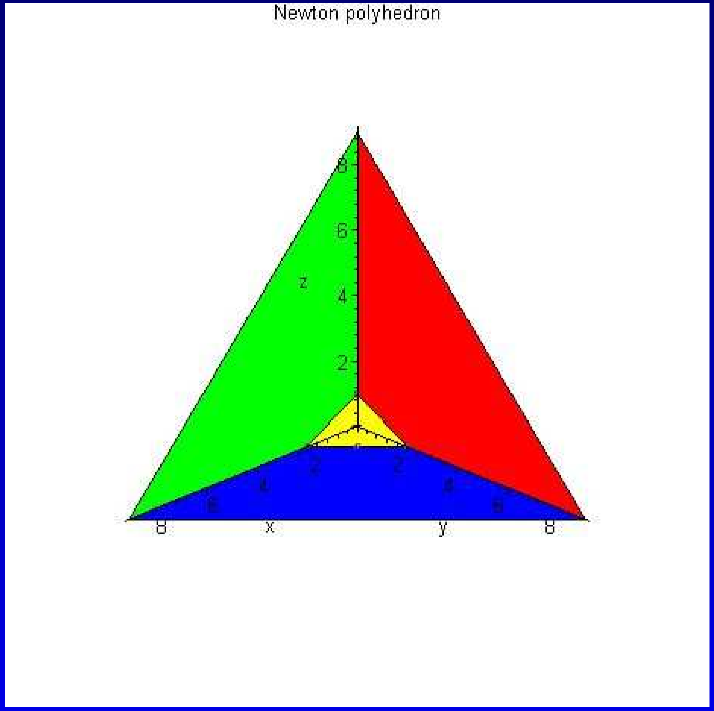
$$+ \sum_{\tau \text{ deg.}} \left(\overline{L_\tau} S_{\Delta\tau} + \sum_{\mathbf{k}} \left(p^{-(\sigma(\mathbf{k})+m(\mathbf{k})s)} \sum_{\alpha \in S} \int_{\alpha + p\mathbb{Z}_p^n} \mid f_\tau + p\tilde{f} \mid^s du_1 \ldots du_n \right) \right)$$

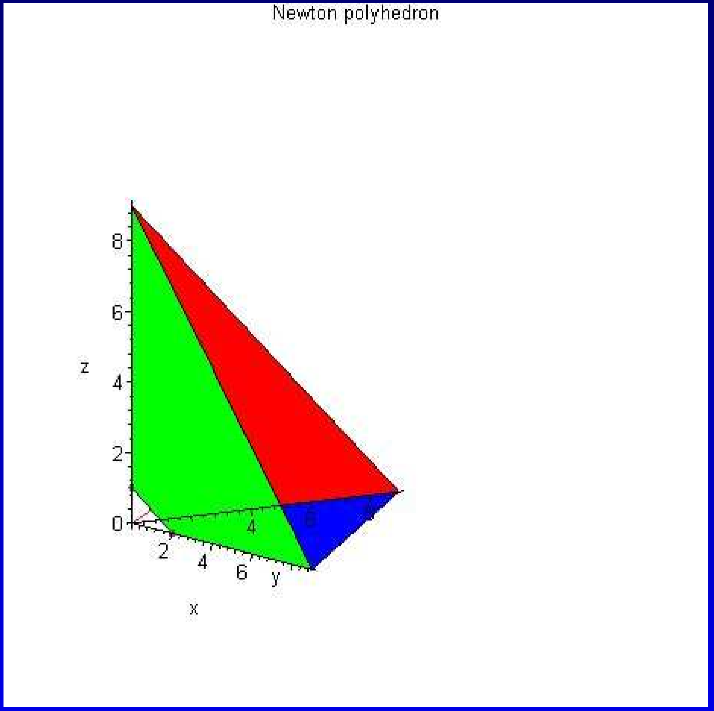
where $f_\tau + p\tilde{f} = p^{m(\mathbf{k})} f$.

Example

Recall the polynomials from earlier:

1. $f(x) = x$
2. $f(x, y, z) = (x - y)^2 + z$





Example

Recall the polynomials from earlier:

1. $f(x) = x$
2. $f(x, y, z) = (x - y)^2 + z$

$$Z(s) = \frac{1 - p^{-1}}{1 - p^{-1}t}$$

for both polynomials, but (1) is non-degenerate while (2) is degenerate!

Future research

- Compare polynomials, both non-degenerate and degenerate, which have the same ILZF.

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- Compare polynomials, both non-degenerate and degenerate, which have the same ILZF.
- Find classes of polynomials for which more can be said about the integral over the singular points in the Newton polyhedron method.

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