IGUSA LOCAL ZETA FUNCTIONS AND THEIR POLES BY THE NEWTON POLYGON METHOD

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ABSTRACT. In her thesis, Hoornaert conjectures that, for certain classes of polynomials, the value $\frac{-1}{t_0}$ would always be a pole of the Igusa local zeta function associated with the polynomial. We study the order of $\frac{-1}{t_0}$ as a pole for specific examples, and conjecture that, if the reasons for the order were understood, a proof or counterexample to Hoornaert's conjecture could be constructed. An introduction to p-adic analysis and the Igusa local zeta function is included as background.

1. Introduction

Our research, conducted at the Mount Holyoke College Mathematics REU during summer 2005, focused on the pole ρ of the Igusa local zeta function. We investigated the actual order of this pole for a specific class of functions, in hopes of better understanding and possibly proving or disproving a conjecture by Kathleen Hoornaert, which states that ρ is always an actual pole of the zeta function for this class of functions. This paper introduces the background to the zeta function, explains some key ideas of p-adic analysis, and goes on to detail our examples and our conjectures about the actual order of the pole. This research was led by Professor Margaret Robinson, and supported by the National Science Foundation.

2. Background

2.1. The p-adic Numbers.

2.1.1. *Structure of $\mathbb{Z}_p$.* In order to understand the Igusa local zeta function, we first need some understanding of the p-adic numbers, as the zeta function is an integer over the p-adic integers. The p-adic number field $\mathbb{Q}_p$ is a way of completing the rational numbers. We express it thus: Let x be a positive integer, and take p prime. We can write a unique p-adic expansion of x in the form

$$x = a_0 + a_1 p + a_2 p^2 + a_3 p^3 + \ldots$$

with $0 \leq a_i \leq p - 1$. This series may be finite or infinite in length.

P-adic numbers can be multiplied, added, subtracted, and divided in their series form, and we use these operations, along with negative powers of p, to build unique p-adic representations of all integers and rational numbers. For example, one can find additive inverses by expressing -1 by the series $(p-1)+(p-1)p+(p-1)p^2+\ldots$ (note that $-1+1 = 0+0p+0p^2+\ldots$=0). This is one way in which infinite series

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Date: July 26, 2005.
RESEARCH SUPPORTED BY NSF GRANT #DMS-0353700

act as p-adic expansions. As division is a closed operation, we build the rationals by dividing integers.

We denote the p-adic integers by $\mathbb{Z}_p$ and the p-adic rationals by $\mathbb{Q}_p$. We often write $p\mathbb{Z}_p$ to refer to all integers with leading term $a = 0$, and $a + p\mathbb{Z}_p$ for the set of all integers with leading term a. We also write $\mathbb{Z}_p^\times$ or $\mathbb{Z}_p \setminus p\mathbb{Z}_p$ for the set of p-adic **units**: that is, p-adic integers with leading term $a_0 \neq 0$. All units are invertible with respect to multiplication: we can invert any unit u by taking $\frac{1}{u}$.

2.1.2. *P-adic Absolute Value.*

Definition 2.1. Let x be an integer and p a prime. We can write $x \neq 0$ uniquely as $x = p^k \hat{x}$, where $gcd(p, \hat{x}) = 1$. Then we define the **p-adic absolute value** of x to be

$$|x|_p = p^{-k}$$

For a rational number $y = \frac{a}{b}$, $|y|_p = \frac{|a|_p}{|b|_p}$.

Example 2.2. If $p = 5$, then $|7|_5 = 1$, $|25|_5 = 5^{-2}$, and $|\frac{7}{25}|_5 = 5^2 = 25$.

To better understand how p-adic absolute value relates to the traditional absolute value, which is denoted p-adically as $|x|_\infty$, remember that the usual absolute value is a measure of a number's distance from 0 on the real line. In contrast, $|x|_p$ measures how far x is from 0 in terms of its divisibility: note that the more times p divides x, the smaller $|x|_p$ becomes. We see that 0 is infinitely divisible by any prime, so $|0|_p = 0$ for all primes. More detail on the construction of $\mathbb{Z}_p$ can be found in Gouvêa's excellent book [3].

2.2. **Integration and the Haar Measure on $\mathbb{Q}_p$.** Given an interval E, we would like to define a measure $M(E)$ on the interval, where M is a map from $\mathbb{Q}_p$ to $\mathbb{Q}^+$. In order to give the measure some nice properties, we use a **Haar measure**, which is defined by the following characteristics:

(1) $M(E) \in \mathbb{R}_+$
(2) $M(\emptyset) = 0$
(3) If $E_1 \cap E_2 = \emptyset$, then $M(E_1 \cup E_2) = M(E_1) \cup M(E_2)$
(4) M is invariant under translation. If $b \in \mathbb{Q}_p$, then $M(b + E) = M(E)$

We normalize the measure so that $M(\mathbb{Z}_p) = 1$. This measure allows us to perform integration over $\mathbb{Z}_p$: we normalize the measure such that $\int_{\mathbb{Z}_p} dx = 1$. We can integrate a function $|f(x)|_p$ over some interval contained in $\mathbb{Z}_p$ by using a **change of variables**, so that if $x = kx' + a$, then $dx = |k|_p dx'$. This allows us to compute a wide range of integrals. There are, however, a number of other methods of integration that allow us to compute more complicated functions, in particular the Igusa local zeta function. For your enjoyment, we will introduce two of these methods, focusing on the Newton polygon method and some of the challenges therein.

3. The Igusa Local Zeta Function

Let $f(x)$ be a polynomial in n variables with integer coefficients. Then the Igusa local zeta function is defined as

$$Z_f(s) = \int_{\mathbb{Z}_p^n} |f(x)|_p^s \, dx$$

where $s \in \mathbb{C}$ for $Re(s) > 0$. It has been shown that $Z_f(s)$ is always a rational function of $t = p^{-s}$ [7]. We are particularly interested in the poles; that is, the values of s such that the denominator vanishes.

3.1. **Calculation of $Z_f(s)$ by the Stationary Phase Formula.** Though the focus of our study was on the Newton polygon method, the Stationary Phase Formula, or SPF, is a useful tool: for many polynomials, especially ones with many non-zero singular points modula p, SPF is an alternate way of computing $Z_f(s)$. Introduced by Igusa in [6], it breaks the integral for $Z_f(s)$ into an integral over three sets:

- the set of $x \in \mathbb{Z}_p^n \setminus p\mathbb{Z}_p^n$, reduced mod p, where $f(x) \not\equiv 0 \bmod p$
- N, the set of $x \in \mathbb{Z}_p^n \setminus p\mathbb{Z}_p^n$, reduced mod p, such that $f(x) \equiv 0 \bmod p$ but not all partial derivatives of $f(x)$ are congruent to 0 mod p
- S, the set of singular points, that is, $x \in \mathbb{Z}_p^n \setminus p\mathbb{Z}_p^n$, reduced mod p, such that $f(x) \equiv 0 \bmod p$ and each partial $\partial/\partial x_i$ is congruent to 0 mod p at x.

The disjoint union of these three sets gives us all of $\mathbb{Z}_p^n$, the interval over which we integrate when we compute $Z_f(s)$. This leads to a way of writing the zeta function in terms of related functions. By convention, we set $t = p^{-s}$. Igusa showed that, for a polynomial $f(x_1, \ldots, x_n)$,

$$Z_f(t) = (p^n - |N|)p^{-n} + (|N| - |S|)p^{-n}t\frac{1-p^-1}{1-p^{-1}t} + \int_S |f(x)|_p^s dx$$

If f has singular points mod p, then we might need to make a change of variables and perform SPF again on the singular integral. It has been conjectured that this method will always, eventually, give a formula for $Z_f(s)$, either by returning $Z_f(s)$ as the singular integral after several iterations of SPF, or will take the form of an infinite series which can be summed, or will eventually terminate. This conjecture is explained in more detail in [1]. For many polynomials, especially degenerate ones, SPF is an effective way to compute the Igusa local zeta function, and it balances the strengths of the Newton polygon method.

4. Calculation of $Z_f(s)$ by the Newton Polygon Method

The Newton Polygon method is another effective way of computing $Z_f(s)$ for certain classes of polynomials, specifically polynomials which are non-degenerate with respect to the faces of their Newton polyhedra. For these polynomials, it offers an explicit formula. In some cases, the zeta function is easier to compute by this method than by others, because there is a smaller set of singular points that affects the calculation.

4.1. **Structure of the Newton Polygon.** Consider a polynomial $f(x)$ on n variables with integer coefficents. We havethat

$$f(x_1, \ldots, x_n) = \sum_{m=(m_1,\ldots,m_n)\in\mathbb{N}^n} k_i x_1^{m_1} \ldots x_n^{m_n}$$

Definition 4.1. The **support** $supp(f)$ is the set of points $\{m\}$.

Definition 4.2. The **Newton polyhedron** $\Gamma(f)$ is the convex hull in the non-negative orthant $(\mathbb{R}^+)^n$ of $\cup_{m\in supp(f)}\{m + \mathbb{R}_+^n\}$. We often talk about the **faces** of a Newton polyhedron, by which we mean the intersections of any supporting hyperplane with the boundary of $\Gamma(f)$, where a supporting hyperplane is defined as

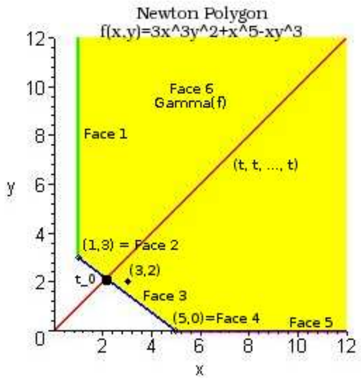

FIGURE 1

a hyperplane that does not intersect the interior of $\Gamma(f)$. In rough terms, this means that any point, line segment, portion of a plane, and so forth on the boundary of $\Gamma(f)$ is a face. We denote a the ith face as τ_i. If τ_i is an $(n-1)$-dimensional face, we call it a **facet**. We also consider the entire $\Gamma(f)$ to be an improper face.

We see that each face τ contains one or more points from $supp(f)$. However, the points in $m \in supp(f)$ come directly from the exponents of the terms of $f(x)$, so we associate to τ a part of f, called $f_\tau(x)$, which we take to be the sum of the terms of $f(x)$ such that $supp(f_\tau) = \tau \cap supp(f)$.

Definition 4.3. Not all faces of $\Gamma(f)$ are compact. Non-compact faces have one or more **directions of recession**. A direction of recession is a vector d such that, for any point $x \in \tau$, $x + \lambda d \in \tau$ for any $\lambda > 0$. When we introduce the cones of Γ, the cones may also have directions of recession.

Example 4.4. Suppose $f(x,y) = 3x^3y^2 + x^5 - xy^3$. Then the points in the support come from the exponents in each term: we get $\{(3,2),(5,0),(1,3)\}$. Observe that for any general n-variable polynomial, the support points will always be in the orthant $(\mathbb{R}^+)^n$, because in each term, for each variable, the exponent must be greater than or equal to 0.

Notice in figure (1) that only two of the support points are actually on the boundary of $\Gamma(f)$. Thus, we have 6 faces: the two points on the boundary, three one-dimensional faces, and one improper face.

4.2. **The Cones.** Based on the structure of $\Gamma(f)$, it is useful to be able to partition the non-negative orthant. We do this by means of what we call the cones of $\Gamma(f)$. To understand the cones, we first need two definitions:

Definition 4.5. Let a be a vector in $(\mathbb{R}^+)^n$. We define the **minimization** of a

$$m(a) = \inf_{x \in \Gamma(f)} \{a \cdot x\}$$

where $\cdot$ denotes the standard dot product.

Definition 4.6. Let a be a vector in $(\mathbb{R}^+)^n$. Then we associate a with the set that gives its minimization; that is, we define the **first meet locus**

$$F(a) = \{x \in \Gamma(f) \mid a \cdot x = m(a)\}$$

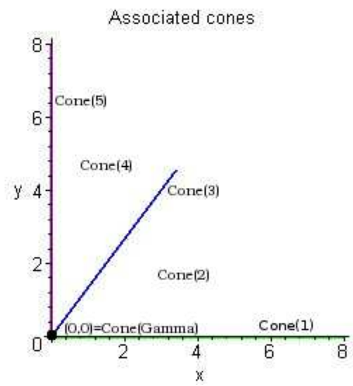

FIGURE 2

Definition 4.7. Finally, the **cone** Δ_τ of a face τ is the set of all $a \in (\mathbb{R}^+)^n$ such that $F(a) = \tau$.

This set of definitions both allows us to partition the non-negative octant, and also gives us a clear relation between the points in Δ_τ and the points in τ, namely that we know the dot product of any vector a with $x \in \tau = m(a)$, and that this dot product is less than that of any point in another cone with a point in $a \cdot y \in \Gamma(f) \setminus \tau > m(a)$.

Example 4.8. Let us consider again our example polynomial, $f(x, y) = 3x^3y^2 + x^5 - xy^3$. Let $a = (0, 3)$. Noting that $m(a)$ must be positive, and that $a \cdot (5, 0) = 0$, we find that $m(a) = 0$ and that $(0, 3) \in F(a)$. If we explore other points, and remember our linear algebra, we see that, if a has the form $a = (0, k)$ then $F(a)$ consists of all the lattice points on τ_5. We conclude that the cone Δ_{τ_5} is the line $\lambda(0, 1)$. Also, for all newton polyhedra, $\Delta_{\Gamma(f)} = (0, \ldots, 0)$. Some experimentation will show that the rest of the cones are as shown in figure (2).

4.3. **An Explicit Formula.**

4.3.1. *The Condition of Non-Degeneracy.* Hoornaert's formula for $Z_f(s)$ relies on the assumption that $f(x)$ is **non-degenerate with respect to all the faces of its Newton polyhedron**.

Definition 4.9. Let τ be a face of $\Gamma(f)$. Note that τ is not necessarily a proper face. Then f is non-degenerate with respect to τ if f_τ has no singular points mod p in the p-adic units, or equivalently if the system

$$\begin{cases} f_\tau(x) \equiv 0 \pmod{p} \\ \partial f_\tau / \partial x_i \equiv 0 \pmod{p} \end{cases}$$

has no solutions in $(\mathbb{Z}_p^\times)^n$ for all $\tau \in \Gamma(f)$.

Note that this condition is slightly different than the conditions for the set S when we perform SPF: in the Newton polygon case, a singular point is not counted if any of its coordinates are congruent to 0 mod p. For example, $(0, 3)$ might be included in S for some functions under SPF, but would not be included in the Newton polygon method.

4.3.2. *The Formula .* Hoornaert's formula uses the idea of summing over all the cones of $\Gamma(f)$ to obtain $Z_f(s)$. Like SPF, this method uses the idea of dividing $\mathbb{Z}_p$ into sets where $f(x) \equiv 0 \mod p$, and sets where it is not. This breakdown is organized by the cones of the Newton polyhedron in a particular way, which we explore more closely in (4.3.3). We use a somewhat different way of counting $|N|$ than we do for SPF.

Definition 4.10. Let τ be a face (not necessarily proper) of $\Gamma(f)$. We denote by N_τ the set of all points $x \in (\mathbb{Z}_p^\times)^n$, reduced modulo p, such that $f_\tau(x) \equiv 0 \mod p$. Observe that, as with the singular points mod p in our discussion of non-degeneracy, only vectors that have units for all of their coordinates are considered: any point with zero mod p as one of its coordinates is excluded. In our work, we frequently used [8] to assist in finding a general formula for N_τ for all primes.

One final piece of notation is needed before presenting the formula:

Definition 4.11. Take $k = (k_1, \dots, k_n) \in \mathbb{R}^n$. We say

$$\sigma(k) = \sum_{i=1}^{n} k_i$$

At last, we have a formula, due to Hoornaert [4]:

$$Z_f(s) = \sum_{\tau \text{ face of } \Gamma(f)} L_\tau S_{\Delta_\tau} \tag{1}$$

where

$$L_\tau = p^{-n}\left((p-1)^n - pN_\tau \frac{p^s - 1}{p^{s+1} - 1}\right) \tag{2}$$

and

$$S_{\Delta_\tau} = \sum_{k \in \mathbf{N}^n \cap \Delta_\tau} p^{-\sigma(k) - m(k)s} \tag{3}$$

In calculating S_{Δ_τ}, it is admittedly difficult to sum over an infinite number of lattice points in the plane. Hoornaert explains exactly how this summation works, and offers a direct formula for the sum S_{Δ_τ}. Take $\tau \in \Gamma(f)$, and suppose $a_1, \dots, a_r, \dots, a_n$ are spanning vectors of the cone Δ_τ. First, if it is necessary, we divide Δ_τ into **simplicial cones**; that is, subcones whose spanning vectors are linearly independent and whose disjoint union is Δ_τ. We denote each subcone Δ_{τ_i} with spanning vectors $a_k, \dots, a_n$. We denote

$$S_{\Delta_{\tau_i}} = \frac{\sum_h p^{\sigma(h)+m(h)s}}{(p^{\sigma(a_k)+m(a_k)s} - 1)\dots(p^{\sigma(a_r)+m(a_r)s} - 1)} \tag{4}$$

where h runs through the set

$$\mathbb{Z}^n \cap \left\{ \sum_{j=1}^{r} \lambda_j a_j | 0 \leq \lambda_j < 1 \right\}$$

4.3.3. *Motivation .* The derivation of the Newton polygon formula for $Z_f(s)$ is quite involved, and is more completely presented in [4], but to demystify this complex and seemingly unmotivated formula, we would like to give a sketch of Hoornaert's proof.

Sketch of Proof. Recall that $Z_f(s) = \int_{\mathbf{Z}_p^n} |f(\vec{x})|_p^s dx$. We are taking an integral over all n-dimensional p-adic integral vectors. Take $x \in \mathbb{Z}_p^n$ and remember that x represents a vector of integers

$$\begin{cases} x_1 = p^{k_1}(a_0 + a_1 p + a_2 p^2 + \ldots) = p^{k_1} u_1 \\ \vdots \\ x_n = p^{k_n}(a_0 + a_1 p + a_2 p^2 + \ldots) = p^{k_n} u_n \end{cases}$$

where u_i is a p-adic unit. If we let $k = (k_1, \ldots, k_n)$ and $u = (u_1, \ldots, u_n)$, then we can write $x = p^k u$. Making this change of variables in $Z_f(s)$, we have

$$dx_i = p^{-k_i} u_i \to dx = p^{-(k_1+\ldots+k_n)} du = p^{-\sigma(k)} du$$

Our new integral then becomes

$$\sum_{k \in \mathbf{N}^n} p^{-\sigma(k)} \int_{(\mathbb{Z}_p \setminus p\mathbb{Z}_p)^n} |f(p^{k_1} u_1, \ldots, p^{k_n} u_n)|^s du$$

Notice that the cones of $\Gamma(f)$ partition the non-negative orthant, and that $k \in \mathbb{N}_p^n$, so we can rewrite our integral over $\mathbb{Z}_p$ as an integral over $\cup_\tau \Delta_\tau \cap \mathbb{N}^n$. When we partition $(\mathbb{R}^+)^n$ into cones, however, we know that the dot product of $k \in \tau$ with any points in τ is at least $m(k)$, which allows us to factor out $p^{m(k)}$ from our integrand, because k is a power to which x is raised. We obtain

$$Z_f(s) = \sum_\tau \sum_{k \in \mathbf{N}^n \cap \Delta_\tau} p^{-\sigma(k)} \int_{(\mathbb{Z}_p \setminus p\mathbb{Z}_p)^n} |p^{m(k)} f(p^{k_1} u_1, \ldots, p^{k_n} u_n)|^s du$$

$$Z_f(s) = \sum_\tau \sum_{k \in \mathbf{N}^n \cap \Delta_\tau} p^{-\sigma(k) - m(k)s} \int_{(\mathbb{Z}_p \setminus p\mathbb{Z}_p)^n} |f_\tau(u) + p\tilde{f}_\tau(u)|^s du$$

We recognize this coefficient: we previously denoted it by S_{Δ_τ} in (3). Because f is non-degenerate, we can show that the remaining integral is equal to our formula for L_τ in (2). Hopefully this explanation gives some justification to the formula for $Z_f(s)$. Hoornaert presents the proof in full detail in [4] if the reader would like a more in-depth understanding.

5. Examination of the Actual Order of the Pole ρ of $Z_f(s)$

5.1. **Background.** A pole is a value where the denominator of a rational function vanishes. Little is known about the actual orders of the poles of the ILZF with respect to a general polynomial. However, we have a clear understanding of what possible poles, or candidate poles, we could have, and for some classes of polynomials we know the actual poles. Using the Newton polygon method, it is straightforward to calculate the candidate poles: our candidates come from the denominators of the S_{Δ_τ} in (4) and L_τ in (2) of the faces of our Newton polygon. However, there can be, and often is, cancellation with the numerator when the zeta function is calculated, and so the order of a candidate pole may be lower than expected, or the candidate may be entirely cancelled out. We studied the actual order of a particular pole, called $/rho$, for a specific class of functions.

Definition 5.1. Consider the unique intersection of the line $(t, t, \ldots, t)$ with the boundary of $\Gamma(f)$. We call this point $t_0 = (t_0, t_0, \ldots, t_0)$. (See figure (1)). Denote by τ_0 the face of $\Gamma(f)$ of lowest dimension that contains the point t_0. As we will show, the complex value ρ, with $Re(\rho) = \frac{-1}{t_0}$, is a candidate pole of $Z_f(s)$. The expected order of ρ as a pole is $\kappa = codim(\tau_0)$. We investigated the order of ρ when $t_0 < 1$.

Theorem 5.2. For any polynomial $f(x)$ meeting the conditions in (4.3.2), $\frac{-1}{t_0}$ is a candidate pole of order κ.

Proof 5.3. Take $a \in (\mathbb{R}^+)^n$. The definition of $m(a)$ implies that

$$(t_0, \ldots, t_0) \cdot a \geq m(a)$$

$$t_0 \sigma(a) \geq m(a)$$

$$\sigma(a) - m(a)\left(\frac{-1}{t_0}\right) \geq 0$$

with equality if and only if $\tau_0 \in F(a)$. Let $a_1, \ldots, a_k$ be the spanning vectors of Δ_{τ_0}. Observe that $k = \kappa = codim(\tau_0)$, because for any face τ, $dim(\tau) + dim(\Delta_\tau) = n$, where n is the full dimension of $\Gamma(f)$. Now, when we compute $Z_f(s)$ by the Newton polygon method, for each a_i, we get a factor $(p^{\sigma(a_i)+m(a_i)s} - 1)$, or $(p^{\sigma(a_i)} - p^{-m(a_i)s})$, in the denominator of S_{Δ_τ}. However, $\tau_0 \in F(a)$, by definition, and so by our previous relation, $\sigma(a_i) - m(a_i)(\frac{-1}{t_0} = 0$. When we let $s = \rho$, all κ factors of the denominator of S_{Δ_τ} vanish. We conclude that ρ is a candidate pole of $Z_f(s)$ of order κ.

Conjecture 5.4. (Hoornaert[4]) Let $f(x_1, \ldots, x_n)$ be a polynomial with integer coefficients, such that f is non-degenerate over $\mathbb{C}$ with respect to the faces of its Newton polyhedron, such that $f(0) = 0$, the origin is a singular point. Suppose $t_0 < 1$. Hoornaert conjectures that ρ is a pole of $Z_f(s)$ for p large enough.

Remark 5.5. For our study, we took "p large enough" to mean that the conjecture should hold true for any prime large enough that neither the coefficients of f or its partial derivatives would cause unwanted vanishing mod p.

Remark 5.6. This conjecture has been proven [4] for the case that no vertex of τ_0 has all its coordinates in the set $\{0, 1, 2\}^n$. We investigated polynomials where one or more vertices of τ_0 fail this condition.

5.1.1. *The Necessity of Having a Singular Point at the Origin* . Consider the following example.

- $f(x, y, z) = x^2 y + x^8 z^2 + z$
  $t_0 = \frac{2}{3} < 1$
  τ_0 has support points $(2, 1, 0), (0, 0, 1)$ with direction of recession $(0, 1, 0)$ and is of dimension 2.
  $Re(\rho) = \frac{-3}{2}$ with expected order of 1.

  Note that the origin is not a singular point because $\partial f / \partial z \neq 0$ at the point $(0, \ldots, 0)$.

  For all values of p, we compute $Z_f(s)$ to be

  $$\frac{p - 1}{p - t}$$

  The value ρ is not the real part of any pole given by the denominator. This example illustrates that the condition of having the origin as a singular point is necessary to the conjecture.

5.2. **Proof that No Examples Exist in Two Variables.**

Theorem 5.7. If f is a polynomial of two variables, and the line $(t,\ldots,t)$ intersects the boundary of $\Gamma(f)$ at $t_0 < 1$, then $(0,0)$ is not a singular point of f, and f does not satisfy the conditions of Hoornaert's conjecture.

Lemma 5.8. If neither $(0,1)$ nor $(1,0)$ is in $supp(\tau_0)$, then $t_0 \geq 1$.

Proof 5.9. (Joanna Miles and Matthew Praegel) Let $f(x,y)$ be a polynomial in 2 variables with integer coefficients. Notice that, if $t_0 < 1$, then τ_0 is one-dimensional, because there are no points $(t,t) \in (\mathbb{N}^+)^2$ with $t < 1$. Consider the supporting hyperplane for τ_0, which we denote by the line l. Let a, b be the x- and y-intercepts of l, respectively, so that l is represented by

$$y - b = \frac{-b}{a}x$$

Suppose $a > 1$ and $b > 1$; we will show that $t_0 \geq 1$. Our supporting hyperplane intersects the line (t,t) when $y = x$. At this intersection, we have

$$x - b = \frac{-b}{a}x$$

$$x = \frac{ab}{b+a} \tag{5}$$

It would be nice to show that both a and b must be at least 2, and not some rational between 1 and 2. This is not difficult, as we must remember that l is the supporting hyperplane of a face of $\Gamma(f)$. We know l must contain at least 2 lattice points: the support points for τ_0. Suppose without loss of generality that $1 \lneq a \lneq 2$. Then, since $a < 2$, l must contain two lattice points: $(0,b)$ and $(1,k)$. But then, by plugging values into equation (5), we have $k = -b(a+1)$. Because a is not an integer, b and k cannot both be integers. Finally, whenever $a, b \geq 2$, we have $t_0 = x \geq 1$, and this proves our lemma.
However, it is still possible that $(0,1)$ or $(1,0)$ is on the hyperplane. (We do not allow $(0,0)$ to be in $supp(f)$.) Assume without loss of generality that $(1,0) \in \tau_0$. We also then have $(0,b) \in \tau_0$, because there are no other positive lattice points on this line. Our function f_{τ_0} taken from these vertices looks like

$$f_{\tau_0} = \alpha x + \beta y^b$$

Herein lies a problem. The partial derivative $\partial f_\tau / \partial x = \alpha$, which by assumption is non-zero. However, Hoornaert's conjecture relies on having $(0,0)$ as a singular point (see (5.1.1) for an example of a function where eliminating this requirement results in $\frac{-1}{t_0}$ not being a pole). Thus, no functions in two variables fit the conditions of the conjecture.

5.3. **Examples.** There follow a number of computed examples, containing cases where the actual order of ρ is the expected order, and examples where it is less than expected. These examples were computed by the programs Polygusa[5] and cdd [2], with assistance from Zeros [8]. Some were also computed by hand by SPF for general p. Unfortunately, we had trouble computing examples in 4 variables for p of any reasonable size, due to the speed and memory constraints of Polygusa. We expect that studying higher dimensions is necessary to grasp the full complexity of the order of the poles. A further study in this area should include a thorough exploration of $Z_f(s)$ of these functions for large primes and dimensions 4 and higher.

Remark 5.10. In [4], Hoornaert proves that $\rho \neq -1$ is a pole **of the expected order** of $Z_f(s)$ if and only if it is a pole of the expected order of $Z_{\tau_0}f(s)$. This reduction of $f(x)$ to $f_{\tau_0}(x)$ allows us to compare the cone structure of related examples, as well as allowing us to work with simplified functions with fewer terms, for which $Z_f(s)$ is frequently easier to compute. Our examples include several families of functions with the same $f_{\tau_0}(x)$.

5.3.1. *Computed examples where ord(ρ) is less than expected.* Recall that, if τ_0 is the smallest face of $\Gamma(f)$ that contains t_0, then the expected order of ρ as a pole of $Z_f(s)$ is $\kappa = codim(\tau_0)$.

- $f(x, y, z, u) = xy + y^3z^2 + z^2$
  $t_0 = \frac{2}{3} < 1$
  τ_0 has support points $(1, 1, 0), (0, 0, 2)$ with no direction of recession and is of dimension 1.
  $Re(\rho) = \frac{-3}{2}$ with expected order of 2.
  For $p = 37$ we compute the denominator of $Z_f(s)$ to be
  $$(-p^3 + t^2)(-p + t)$$
  The pole ρ comes from the term $(-p^3 + t^2)$ and has actual order 1.
  For "many" primes p, we compute
  $$Z_f(s) = \frac{(p-1)(-p^3+t)}{(-p^3+t^2)(-p+t)}$$
  and so the order of ρ is 1 for these p. However, there is no regular formula for $|N_\tau|$ for all p. Even for some varying $|N_\tau|$, however, the zeta function remains unchanged, and may be the same for all primes. This is a question for further investigation.

- $f(x, y, z) = xy + z^2$
  $t_0 = \frac{2}{3} < 1$
  τ_0 has support points $(1, 1, 0), (0, 0, 2)$ with no direction of recession and is of dimension 1.
  $Re(\rho) = \frac{-3}{2}$ with expected order of 2.
  For all p we compute $Z_f(s)$ to be
  $$\frac{-(-p^3+t)(p-1)}{(-p^3+t^2)(-p+t)}$$
  The pole ρ comes from the term $(-p^3 + t^2)$ and has actual order 1 for all p.

- $f(x, y, z) = xy - x^2y + y^2z + x^2 + z^2$
  $t_0 = \frac{2}{3} < 1$
  τ_0 has support points $(1, 1, 0), (0, 0, 2)$ and is of dimension 1.
  $Re(\rho) = \frac{-3}{2}$ with **expected order of 2**.
  For $p = 37, 43, 47$ we compute the denominator of $Z_f(s)$ to be
  $$p(-p^3 + t^2)(-p + t)$$
  The pole ρ comes from the term $(-p^3 + t^2)$ and has **actual order 1**.

- $f(x, y, z) = x^2 + yz + z^2xy$
$t_0 = \frac{2}{3} < 1$
τ_0 has support points $(2, 0, 0), (0, 1, 1)$ and is of dimension 1.
$Re(\rho) = \frac{-3}{2}$ with **expected order of 2**.
For $p = 37, 43, 47$ we compute the denominator of $Z_f(s)$ to be
$$p^2(-p^3 + t^2)(-p + t)$$
The pole ρ comes from the term $(-p^3 + t^2)$ and has **actual order 1**.
In general, we compute
$$Z_f(s) = \frac{(p-1)(p^5 + p^3 - p^3t - p^2t + t^3 - t^2)}{p^2(-p^3 + t^2)(-p + t)}$$
and so the order of ρ is 1 for all p.
Note that the previous several examples form a family under the reduction in (5.10).

- $f(x, y, z) = x^3 + yz + z^2xy$
$t_0 = \frac{3}{4} < 1$
τ_0 has support points $(0, 1, 1), (3, 0, 0)$ with no direction of recession and is of dimension 1.
$Re(\rho) = \frac{-4}{3}$ with **expected order of 2**.
For $p = 37, 47$ we compute the denominator of $Z_f(s)$ to be
$$p^2(-p^4 + t^3)(-p + t)$$
The pole ρ comes from the term $(-p^4 + t^3)$ and has actual order 1.
In general, we compute
$$Z_f(s) = \frac{(p-1)(p^6 + p^4 - p^4t + p^3t^2 - p^3t - p^2t^2 + t^4 - t^3)}{p^2(-p^4 + t^2)(-p + t)}$$
and so the order of ρ is 1 for all p.
In fact, this works also for the first term equal to x^2, x^3, x^4 and so on. We conjecture that the order is 1 for all $f(x, y, z) = x^a + yz + z^2xy$ and that $t_0 = \frac{a}{a+1}$. We claim that $|N_{\Gamma(f)}| = (p-1)^2 - (p-1)$ for all p.

- $f(x, y, z, u) = x^2y + x^8z^2 + zu$
$t_0 = \frac{2}{3} < 1$
τ_0 has support points $(2, 1, 0, 0), (0, 0, 1, 1)$ with direction of recession $(0, 1, 0, 0)$ and is of dimension 2.
$Re(\rho) = \frac{-3}{2}$ with **expected order of 2**.
For $p = 7, 11, 17$ we compute the denominator of $Z_f(s)$ to be
$$(-p^3 + t^2)(-p + t)$$
The pole ρ comes from the term $(-p^3 + t^2)$ and has **actual order 1**.

- $f(x, y, z, u) = x^2y^2z + x^3z^2u^3 + xy + yzu + xzu + zu$
$t_0 = \frac{1}{2} < 1$
τ_0 has support points $(1, 1, 0, 0), (0, 0, 1, 1)$ with **no** direction of recession and is of dimension 1.
$Re(\rho) = -2$ with **expected order of 3**.
For $p = 17, 23$ we compute the denominator of $Z_f(s)$ to be
$$p^3(-p^2 + t)(-p + t)$$

The pole ρ comes from the term $(-p^2+t)$ and has **actual order 1**.

- $f(x,y,z,u) = xy + zu$
$t_0 = \frac{1}{2} < 1$
τ_0 has support points $(1,1,0,0),(0,0,1,1)$ with **no** direction of recession and is of dimension 1.
$Re(\rho) = -2$ with **expected order of 3**.
For all p we compute $Z_f(s)$ to be
$$\frac{(1+p)(p-1)^2)}{(-p^2+t)(-p+t)}$$
The pole ρ comes from the term $(-p^2+t)$ and has **actual order 1**. This polynomial is equal to f_{τ_0} of the previous function.
- $f(x,y,z,u) = z^2u^2 + xy + y^3z + yzu + u^3$
$t_0 = \frac{2}{3} < 1$
τ_0 has support points $(1,1,0,0),(0,0,2,2)$ with **no** direction of recession and is of dimension 1.
$Re(\rho) = \frac{-3}{2}$ with **expected order of 3**.
For $p = 17, 23$ we compute the denominator of $Z_f(s)$ to be
$$(t^4+p^3t^2+p^6)(-p^3+t^2)^2(-p+t)$$
The pole ρ comes from the term $(-p^3+t^2)^2$ and has **actual order 2**.
- $f(x,y,z,u) = z^2u + xy + y^3z + yzu + u^2$
$t_0 = \frac{4}{7} < 1$
τ_0 has support points $(1,1,0,0),(0,0,2,1),(0,0,0,2)$ with **no** direction of recession and is of dimension 2.
$Re(\rho) = \frac{-3}{2}$ with **expected order of 2**.
For $p = 23$ we compute the denominator of $Z_f(s)$ to be
$$(-p^7+t^4)(-p+t)$$
The pole ρ comes from the term $(-p^7+t^4)$ and has **actual order 1**.
- $f(x,y,z,u) = x^2y + y^2xz^3 + u^3$
$t_0 = \frac{3}{4} < 1$
τ_0 has support points $(1,0,3,0),(2,1,0,0),(0,2,0,0),(0,0,0,3)$ with **no** direction of recession and is of dimension 3.
$Re(\rho) = \frac{-3}{2}$ with **expected order of 2**.
For $p = 23$ we compute the denominator of $Z_f(s)$ to be
$$(t^6+23^8)(t^6-23^8)(-p+t)$$
The pole ρ comes from the term (t^6-23^8) and has **actual order 1**. We should investigate the unusual form of this denominator.

5.4. **Computed examples where $ord(\rho)$ is the same as expected.**

- $f(x,y,z,u) = x^2y^2z + x^3z^2u^3 + xy + yzu + xzu + zu$
$t_0 = \frac{1}{2} < 1$
τ_0 has support points $(1,1,0,0),(0,0,1,1)$ with no direction of recession and is of dimension 1.
$Re(\rho) = -2$ with expected order of 3.; an

For $p = 17, 23$ we compute the denominator of $Z_f(s)$ to be

$$p^2(-p^2 + t)(-p + t)$$

The pole ρ comes from the term $(-p^2 + t)$ and has actual order 1.

- $f(x, y, z) = -x^2y - y^2z + x^2 + z^2$
  $t_0 = \frac{4}{5} < 1$
  τ_0 has support points $(2, 0, 0), (0, 0, 2), (0, 2, 1)$ with no direction of recession and is of dimension 2.
  $Re(\rho) = \frac{-5}{4}$ with expected order of 1.
  For $p = 47, 53$ we compute the denominator of $Z_f(s)$ to be

  $$(-p^5 + t^4)(-p + t)$$

  The pole ρ comes from the term $(-p^5 + t^4)$ and has actual order 1.
- $f(x, y, z) = x^2y + x^8z^2 + yz$
  $t_0 = \frac{12}{13} < 1$
  τ_0 has support points $(2, 1, 0), (8, 0, 2), (0, 1, 1)$ with no direction of recession and is of dimension 2.
  $Re(\rho) = \frac{-13}{12}$ with expected order of 1.
  For $p = 37, 43$ we compute the denominator of $Z_f(s)$ to be

  $$(-p^13 + t^12)(-p + t)$$

  The pole ρ comes from the term $(-p^13 + t^12)$ and has actual order 1.
- $f(x, y, z, u) = x^2y^2z + xu^3z^2 + xy + yzu$
  $t_0 = \frac{4}{5} < 1$
  τ_0 has support points $(1, 0, 2, 3), (1, 1, 0, 0), (0, 0, 1, 1)$ with direction of recession $(0, 0, 1, 0)$ and is of dimension 3.
  $Re(\rho) = \frac{-13}{12}$ with expected order of 1.
  For $p = 17$ we compute the denominator of $Z_f(s)$ to be

  $$p^3(-p^4 + t^3)(-p^5 + t^4)(-p + t)$$

  The pole ρ comes from the term $(-p^5 + t^4)$ and has actual order 1.
- $f(x, y, z, u) = x^2y + z^3u + u^2$
  $t_0 = \frac{6}{7} < 1$
  τ_0 has support points $(0, 0, 3, 1), (2, 1, 0, 0), (0, 0, 0, 2)$ with direction of recession $(0, 1, 0, 0)$ and is of dimension 3.
  $Re(\rho) = \frac{-7}{6}$ with expected order of 1.
  For $p = 23$ we compute the denominator of $Z_f(s)$ to be

  $$(-p^7 + t^6)(-p + t)$$

  The pole ρ comes from the term $(-p^7 + t^6)$ and has actual order 1.
- $f(x, y, z) = xy + yz + xz$
  $t_0 = \frac{2}{3} < 1$
  τ_0 has support points $(1, 0, 1), (0, 1, 1), (1, 1, 0)$ with no direction of recession and is of dimension 2.
  $Re(\rho) = \frac{-3}{2}$ with expected order of 1.
  For $p = 37, 47$ we compute the denominator of $Z_f(s)$ to be

  $$(-p^3 + t^2)(-p + t)$$

The pole ρ comes from the term $(-p^3+t^2)$ and has actual order 1.

5.5. **Further Conjectures Regarding the Poles.** We have been investigating cases where $t_0 < 1$, partly in search of counterexamples to Hoornaert's conjecture, but primarily in an effort to understand when the order of ρ as a pole of $Z_f(s)$ is less than expected. The above examples show that the order is less than expected for many functions. We have noted the actual and expected orders.

Observation. For several examples, specifically $f(x,y,z) = xy+z^2$ and $g(x,y,z,u) = xy+zu$, we explored the cones of their Newton polygons, and found a direct correlation between the cones and the orders of ρ in S_{Δ_τ}. For the first example, we found that the order of ρ as a pole of $S_{\Delta_{\tau_i}}$ was exactly equal to the to the number of spanning vectors that Δ_{τ_i} shares with Δ_{τ_0}. In the second case, this held true except when $S_{\Delta_{\tau_i}}$ contained the full expected order of ρ, in which case it shared all 4 spanning vectors, but only contained 3 copies of ρ as a pole. However, these cones were still 3-dimensional, non-simplicial cones, and 3 is the maximum possible order of any pole for a 3-variable function. We expect that this pattern holds in some general form for all functions. It is interesting to note that the actual order in the second example was 1, instead of 3, so there may be a corellation between the highest number of shared spanning vectors and the difference between the actual order and the expected order. More likely, however, this property follows directly from our definition of τ_0. In this case, it should be straightforward to prove a general case giving the order of ρ as a pole of each $S_{\Delta_{\tau_i}}$.

Conjecture on Cancellation. One focus has been an effort to understand what leads to the cancellation that causes the order to be less than expected. We expected that all necessary cancellation might come from

$$\sum_{\tau_j} S_{\Delta_{\tau_j}} L_{\tau_j}$$

where τ_j is a face such that $S_{\Delta_{\tau_j}}$ has the order of ρ as a pole greater than the actual order in $Z_f(s)$. However, this method did not work for all examples: for instance, when $f(x,y,z) = xy+z^2$, there was no cancellation from adding the terms of high order. A more subtle form of cancellation is taking place. We also investigated the sum

$$\sum_{\tau_k} S_{\Delta_{\tau_k}} L_{\tau_k}$$

where $\Delta_{\tau_k} \in \bar{\Delta}_{\tau_0}$. In some cases, this provided some of the necessary cancellation, but this process failed for $g(x,y,z,u) = xy+zu$. It is not clear that any particular sum will provide the needed cancellation, but we expect that the cone structure of Δ_{τ_0} and the surrounding cones is closely related to the actual order of ρ as a pole of $Z_f(s)$.

Observation on Proving or Disproving the Conjecture. We were unable to find a counterexample to Hoornaert's conjecture. However, part of our motivation in studying the actual order of ρ as a pole was to better understand the possibilities for finding a counterexample. We believe that, if the conjecture is false, a deeper understanding of what causes the order to be less than expected may allow one to create an example where all ρ poles cancel in the final zeta function, thus constructing a counterexample. We also suspect that, if a counterexample exists, there may be a minimum dimension in which counterexamples can be found. This minimum dimension may be higher than the dimensions we studied, and investigating functions with 5 or more variables may be necessary to better understand the actual order of the pole.

6. Acknowledgements

There are many people who made this research possible, and who offered their support during the 2005 REU at Mount Holyoke. I would like to thank Margaret Robinson, first and foremost, for her leadership, advice, teaching, and friendly help. Without her, this summer would never have taken off. I am also in the debt of my groupmates Adrienne Rau, Ricardo Portilla, Matthew Praegel, Rosica Dineva, and Rex Chung for their encouragement, for our many shared hours of problem-solving, and for making the research more fun. Of course I owe a great deal of thanks to the National Science Foundation for supporting the REU program and making our research possible. Finally, I would like to thank my advisor, Michael Henle, for his wisdom and support that helped me get here in the first place.

References

1. Mary Fleischli, *Local zeta functions associated with polynomials of two and three variables*, Honors paper for the Mount Holyoke College Department of Mathematics, May 1993.
2. K. Fukuda, *The source code of the program cdd*, http://www.ifor.math.ethz.ch/ fukuda/cdd_home/cdd.html, 1997.
3. Fernando Q. Gouvêa, *p-adic numbers*, 2 ed., Springer, 1997.
4. Kathleen Hoornaert, *Newton polyhedra and igusa's local zeta function*, Ph.D. thesis, Katholieke Universiteit Leuven, 2002.
5. Kathleen Hoornaert and Davy Loots, *Polygusa 0.5.7*, http://wis.kuleuven.be/algebra/kathleen/program.htm, July 2002, A program that provides information on the Newton polyhedra and Igusa local zeta functions for specific polynomials.
6. J.-I. Igusa, *A stationary phase formula for p-adic integrals and its applications*, Algebraic geometry and its applications, Springer-Verlag, New York, 1994, pp. 175–94.
7. ———, *An introduction to the theory of local zeta functions*, Studies in Advanced Mathematics **14** (2000).
8. Joanna Miles, *Zeros1.2*, July 2005, A program written to assist in computing the zeros of a function modulo a prime, for use in the calculation of zeta functions.