The Mysterious Actual Order of the Pole ρ of the Igusa Local Zeta Function

Exploration of a Conjecture

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- For other classes of functions, very little is known about ρ and its order.
- We studied one of the latter classes.

p-adic Numbers

What is a p-adic integer?

- Fix p prime.
- If m is a p-adic integer, $\exists$ a unique p-adic expansion

$$m = a_0 + a_1 p + a_2 p^2 + a_3 p^3 + \ldots$$

with $0 \leq a_i \leq p - 1$.

- $\mathbb{Z}_p$: the set of all p-adic integers.

The Igusa Local Zeta Function

Let $f(x)$ be a polynomial in n variables with integer coefficients.

$$Z_f(s) = \int_{\mathbb{Z}_p^n} |f(x)|_p^s \, dx$$

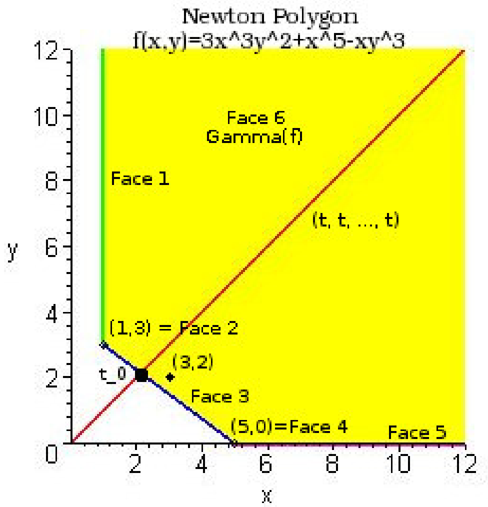
where $s \in \mathbb{C}$ for $Re(s) > 0$.

$Z_f(s)$ is always a rational function of $t = p^{-s}$ (Igusa, 1975).

Structure of the Newton Polygon

Newton polygon $\Gamma(f)$ for
$f(x, y) = x^3y^2 + x^5 - xy^3$:

- Support points: ~~$(3, 2)$~~, $(5, 0)$, $(1, 3)$



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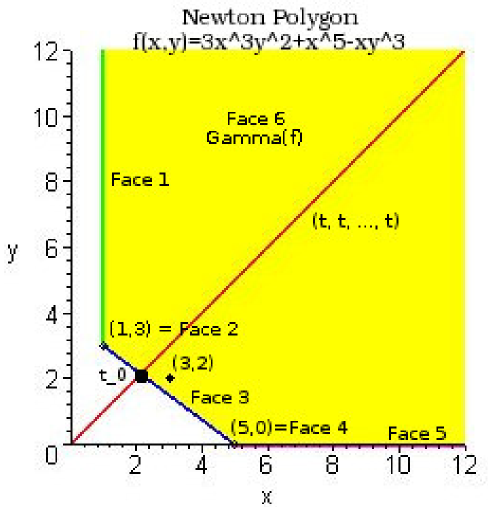

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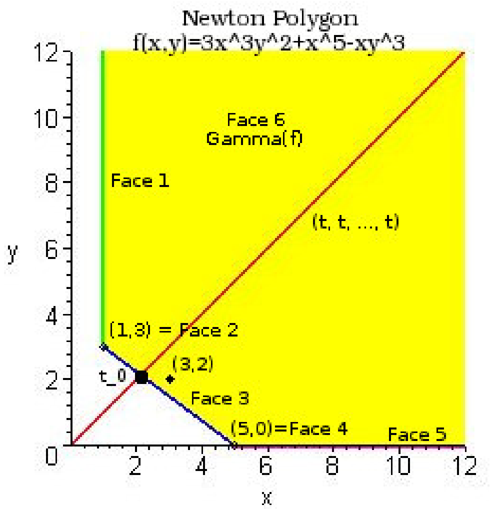

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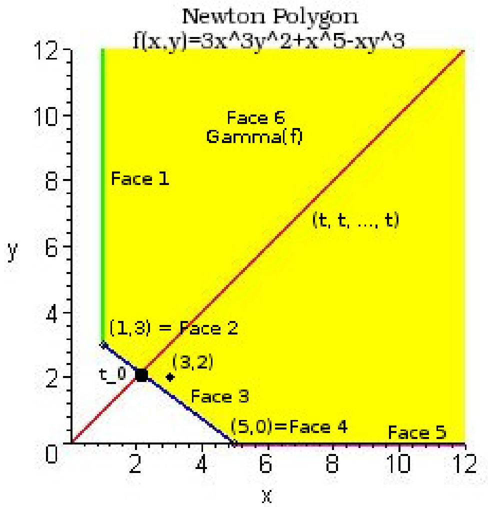

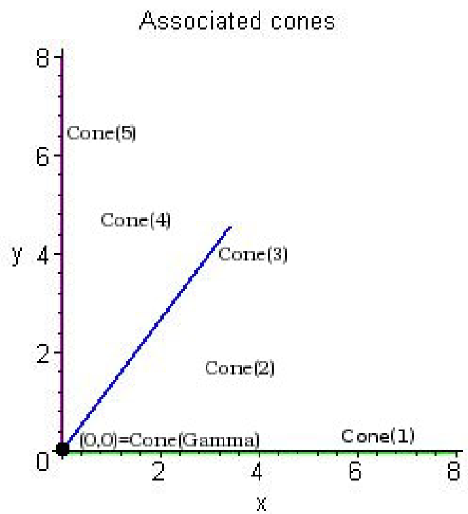

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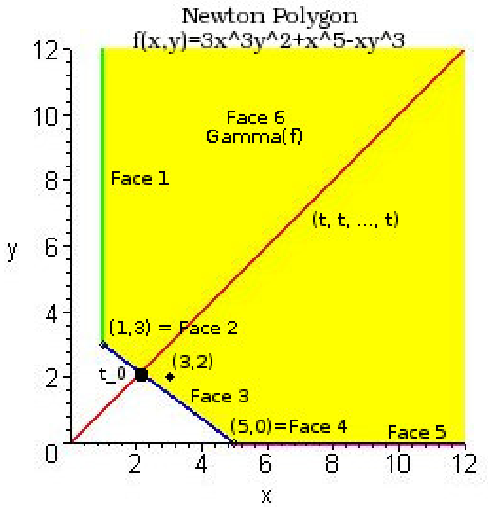

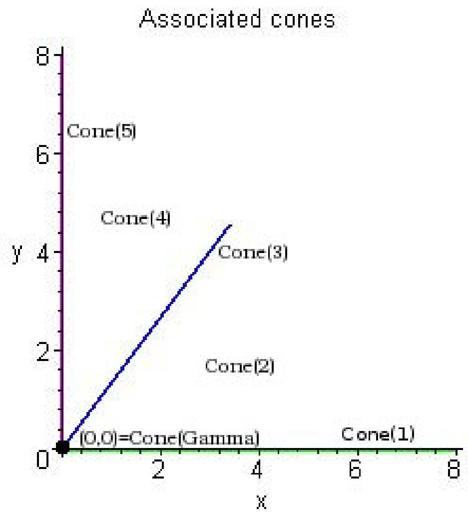

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- Each face has associated polynomial f_τ.
- Cones: associated to the faces, form a partition of the first quadrant.
- The cones give us regions of $\mathbb{Z}_p$ for integration.

Calculation of $Z_f(s)$ Using the Newton Polygon Method

Let $f(x)$ be non-degenerate: it has no singular points mod p in the p-adic units for any $f_\tau(x)$.

$$Z_f(s) = \sum_{\tau \text{ face of } \Gamma(f)} L_\tau S_{\Delta_\tau}$$

where

$$L_\tau = p^{-n}\left((p-1)^n - p|N_\tau|\left(\frac{p^s-1}{p^{s+1}-1}\right)\right)$$

$$S_{\Delta_\tau} = \sum_{k \in \mathbf{N}^n \cap \Delta_\tau} p^{-\sigma(k)-m(k)s}$$

$\sigma(k) = \sum_{i=1}^n k_i$
$m(a) = inf_{x\in\Gamma(f)}\{a \cdot x\}$

N_τ contains all x such that $f_\tau(x) \equiv 0 \bmod p$

Our Candidate Poles and Expected Orders

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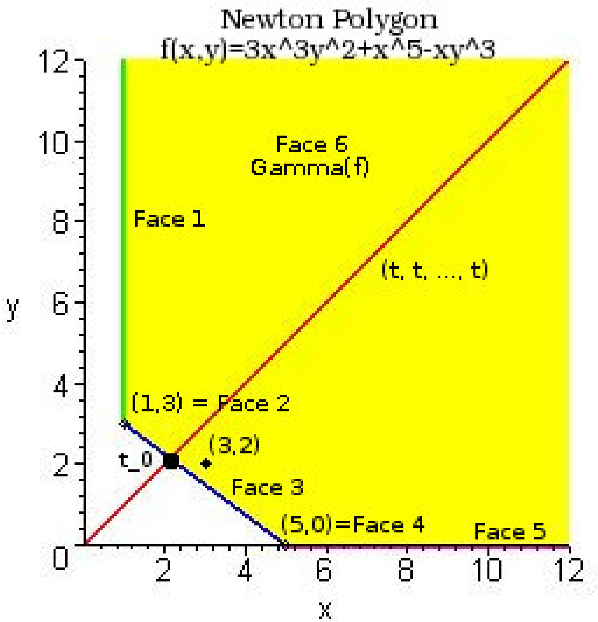
The expected order of a candidate pole (other than -1) is the maximal order of that pole from all S_{Δ_τ}.

Of course, when we add terms, some of these poles disappear…

Our Favorite Candidate Pole: t_0



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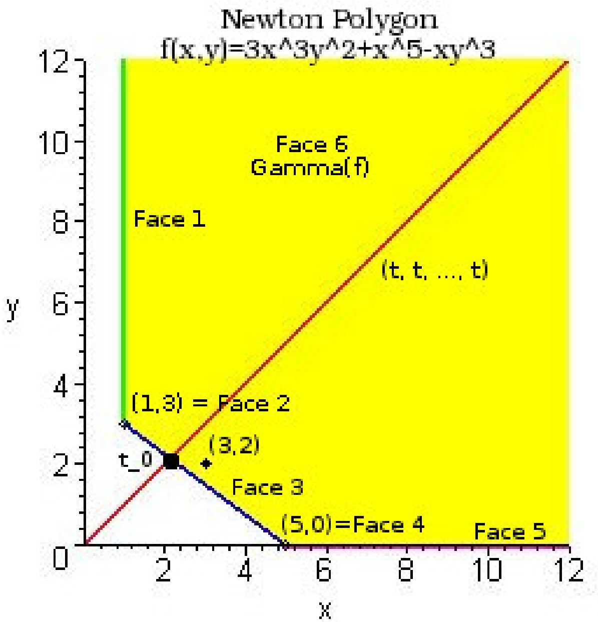

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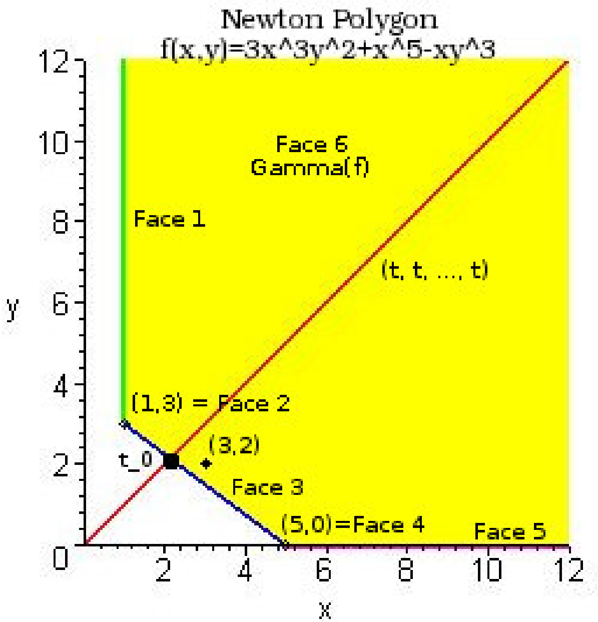

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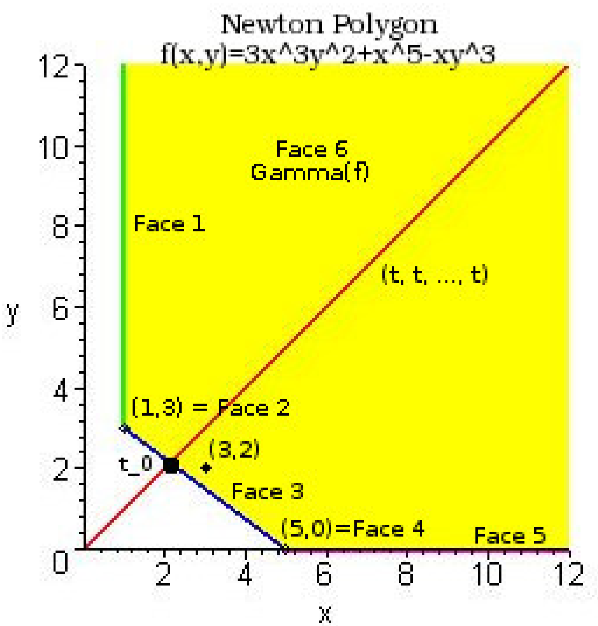

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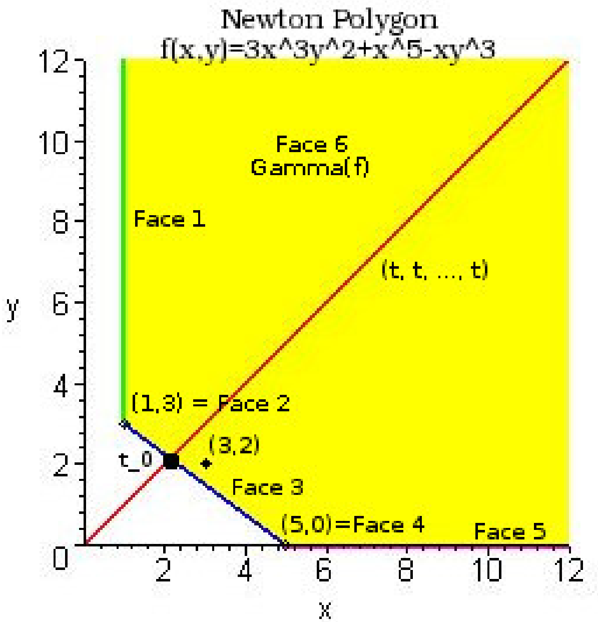

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- $\kappa = codim(\tau_0)$: The expected order of ρ.

When t_0 Gives an Actual Pole

Take $f(x)$ such that:

- $f(x)$ non-degenerate wrt the faces of $\Gamma(f)$
- $f(0) = 0$
- $\partial f / \partial x_i(0) = 0 \;\; \forall i$

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If $t_0 > 1 \longrightarrow \rho$ is an actual pole of the expected order, κ, for p large enough. (PROVEN)

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If $t_0 < 1$ and no vertex of $\tau_0 \in \{0, 1, 2\}^n \longrightarrow \ \rho$ is an actual pole for p large enough. (PROVEN)

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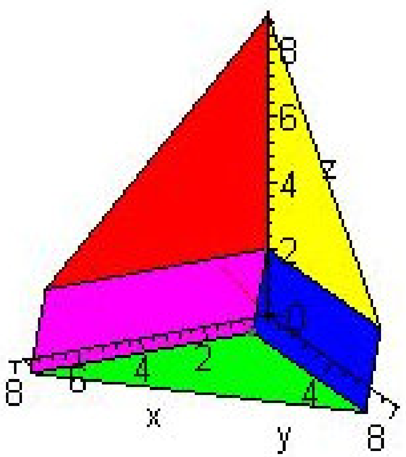
This is the class of examples we studied. There are no known counterexamples to the conjecture.

Cancellation and Actual Orders

- **Almost nothing** is known about the order of ρ as a pole for this class of functions.
- Often, actual order < expected order. Sometimes it is much less.
  Our Conjecture: If we understood what causes low order, we could force more cancellation, and eliminate **all** terms that give ρ as a pole. This would provide a counterexample.
- We studied examples in 3 and 4 variables and examined the actual order of ρ as a pole.

An Example

$$f(x,y,z) = xy + z^2$$



- $t_0 = \frac{2}{3} < 1$
- τ_0 is a line between $(1,1,0)$ and $(0,0,2)$, with dimension 1.
- $Re(\rho) = \frac{-3}{2}$ and expected order is 2.
- $Z_f(s) = \frac{-(-p^3+t)(p-1)}{(-p^3+t^2)(-p+t)}$
- Actual order of ρ is 1.
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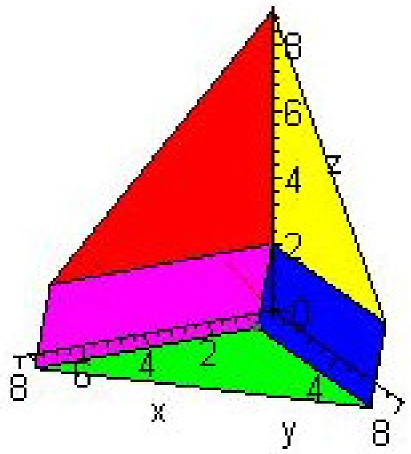

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- If we sum all $S_{\Delta_\tau} L_\tau$, where ρ has order 2 in S_{Δ_τ}, we get no cancellation.

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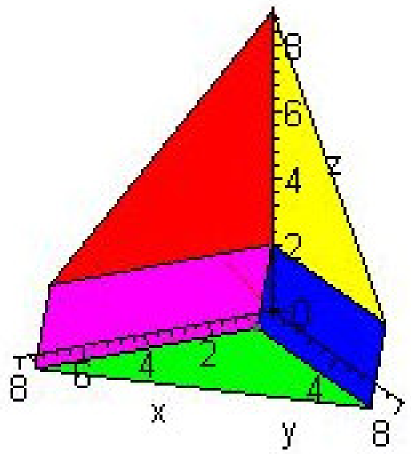

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- When we sum S_{Δ_τ} for all $\Delta_\tau \subset \overline{\Delta}_{\tau_0}$, we **do** have cancellation!

What We've Learned

- We still know little about the orders of ρ as a pole for our class of functions.
- The cancellation is not "nice": it does not come just from the S_{Δ_τ} terms, nor does it usually have an obvious algebraic source.
- Studying dimensions 4 and higher will give opportunities for more cancellation, and will provide the most interesting examples.

Conclusion: Our Conjectures

- Is Hoornaert's conjecture true?
- If not, can we find a counterexample by studying the sources of cancellation in $Z_f(s)$?
- We expect that the actual order of ρ is related to the structure of $\Gamma(f)$, particularly to Δ_{τ_0} and the surrounding cones.
- If counterexamples exist, what is the example of lowest dimension?
- We know ρ is always a candidate pole. The question: Can enough terms cancel that ρ is not an actual pole?

Thanks

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