TECHNIQUES FOR COMPUTING THE IGUSA LOCAL ZETA FUNCTION

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Abstract. We have investigated the Igusa local zeta function using two varieities of methods: Stationary Phase Formula and the Newton polygon method. We then looked at some examples as to the relationship between the order of a pole of the zeta function and the contributing cones from the Newton Polygon.

1. Introduction

We are studying p-adic numbers with their application to the Igusa local Zeta function. The Igusa Zeta function is of interest because it connects to Number thoery where it can be used for determining the Poincaré series. We concentrated on two main techniques, Stationary Phase Formula and the Newton polyhedra method, for determining the Zeta function. With the Newton polyhedra method, we looked into the roots of the Igusa local Zeta function. They are of interest because they represent where the otherwise rational function becomes undefined for specific complex values of s. Specifically we looked at the times when a root of the Zeta function has less multiplicity than expected in hopes that a counterexample to Kathleen Hoornaert's conjecture would be found. We were unable to devise such an example, but desire to find the classes of functions to which the expected order of the pole is less than expected.

2. p-adic numbers

Before we can introduce the Zeta function, we need to start with background on p-adic numbers.

Definition 2.1. A *p-adic number* x, can be represented in the form:

$$a_1p^k + a_2p^{k+1} + a_3p^{k+2} + a_4p^{k+3} + ...$$

where $a_i \in \frac{\mathbb{Z}}{p\mathbb{Z}}$.

Date: July 2005.

Research supported by NSF grant #DMS-0353700.

P-adic numbers can be multiplied, subtracted and divided all similar to the real numbers. They are derived from the rational numbers by using the limit of all cauchy sequence in the same method that the real numbers are derived from the rational numbers. The set of p-adic numbers forms a field. [?]

Definition 2.2. The *set of p-adic numbers* is $\mathbb{Q}_p$.

Each rational number can be reperesented by a p-adic number so $\mathbb{Q} \subset \mathbb{Q}_p$.

Similarly, we have the p-adic integers.

Definition 2.3. The *set of p-adic integers* is

$$\left\{ a_1 + a_2p^1 + a_3p^2 + a_4p^3 + \ldots | a_i \in \frac{\mathbb{Z}}{p\mathbb{Z}} \right\}$$

They are represtented by $\mathbb{Z}_p$.

With the definition of p-adic numbers also comes a new definition for absolute value.

Definition 2.4. The *p-adic absolute value* of a number

$$x = a_1p^k + a_2p^{k+1} + a_3p^{k+2} + a_4p^{k+3} + \ldots$$

denoted as $|x|_p$, as where $a_i \in \frac{\mathbb{Z}}{p\mathbb{Z}}$ is defined to be

$$|x|_p = p^{-k}$$

From definition,

$$|0|_p = 0$$

For the ease of notation, from now on whenever we refer to the p-adic absolute value in this paper, we will use the normal absolute value symbol.

$$|x|_p = |x|$$

From the definition of absolute value, the p-adic numbers not only exhibit the triangle inequality, but they also take on a ultra-metric property.

Proposition 2.5. *The p-adic numbers have the* ultra-metric *property. That is,*

$$|x + y| \leq \ max\ (|x|, |y|)$$

From this, the triangle inequality is implied.

From these definitions, we can now introduce the integration over the p-adic numbers.

3. p-adic Integration

The set of p-adic number and the set of p-adic integers are locally compact. From this, the Haar measure is developed on $\mathbb{Q}_p$.

Remark 3.1. The Haar has the following properties for $E = \alpha + p^e\mathbb{Z}_p$ and $m(E) \in \mathbb{R}^+ \bigcup 0$:

$$m(E) \geq 0 \tag{1}$$
$$m(\emptyset) = 0 \tag{2}$$
$$\text{If } E_1 \bigcap E_2 = \emptyset, \text{ then } m(E_1 \bigcup E_2) = m(E_1) + m(E_2) \tag{3}$$
$$m(\beta + E) = m(E) \text{ for } \beta \in \mathbb{Q}_p \tag{4}$$
$$m(\mathbb{Z}_p) = 1, \int_{\mathbb{Z}_p} dx = 1 \tag{5}$$

The fourth property is called translation invariance of $\mathbb{Q}_p$.

Theorem 3.2. *The Haar measure is unique up to a positive real number*

The proof of this is ilistrated in Tate's Thesis. In part of the remark,

$$m(\mathbb{Z}_p) = 1$$

, fixing the positive real number in which the rest of the set can be measured from.

Lemma 3.3. *From this information about the Haar measure, we also find that*

$$m(p\mathbb{Z}_p) = p^{-1}$$

Proof. From remark 3.1.5, we know that

$$1 = \int_{\mathbb{Z}_p} dx = \sum_{\beta=0}^{p-1} \int_{\beta + p\mathbb{Z}_p} = \sum_{\beta=0}^{p-1} \int_{p\mathbb{Z}_p} = p \int_{p\mathbb{Z}_p}$$

This shows that $m\left(p\mathbb{Z}_p\right) = p^{-1}$ □

Using a similar argument, we can also show that

$$m(p^e\mathbb{Z}_p) = p^{-e}$$

4. IGUSA LOCAL ZETA FUNCTION

From the information about integration over $\mathbb{Z}_p$ we can now understand the Igusa local zeta function.

Definition 4.1. The *Igusa local zeta function*, $Z(s)$ is defined by

$$Z(s) = \int_{\mathbb{Z}_p^n} |f(x)|^s \, dx$$

Using the fact that $\int_{Z_p^n} dx = 1$ from the definition of the Haar measure, we can now compute some simple Zeta functions.

Example 4.2. For $f(x) = x$, we can compute the zeta function using a substitution method.

$$\int_{\mathbb{Z}_p} |x|^s dx = \int_{p\mathbb{Z}_p} |x|^s dx + \sum_{\alpha=1}^{p-1} \int_{\alpha+pZ_p} |x|^s dx = \int_{p\mathbb{Z}_p} |x|^s dx + \sum_{\alpha=1}^{p-1} \int_{\alpha+pZ_p} 1dx$$

This occurs because $|x|^s = 1$ when $x \in \alpha + p\mathbb{Z}_p$. Therefore since

$$m\left(\alpha + p\mathbb{Z}_p\right) = m\left(p\mathbb{Z}_p\right) = p^{-1}$$

and there are $(p-1)$ of these sets, the zeta function reduces to

$$Z(s) = \int_{p\mathbb{Z}_p} |x|^s dx + (p-1)p^{-1}$$

In order to determine $\int_{p\mathbb{Z}_p} |x|^s dx$, we can use a change of variables technique. Let $x = pu$ so therefore $dx = p^{-1}du$. Thus we can consider

$$Z(s) = p^{-1} \int_{\mathbb{Z}_p} |pu|^s du + (p-1)p^{-1} = p^{-1}p^{-s} \int_{\mathbb{Z}_p} |u|^s du + \left(1 - p^{-1}\right)$$

Here we obtain a recursion since $\int_{\mathbb{Z}_p} |u|^s du = Z(s)$, so we can deduce that

$$Z(s) = \frac{(1 - p^{-1}}{1 - p^{-1}p{-}s}$$

Another important series related to number theory is the Poincaré series.

Definition 4.3. The *Poincaré series* is defined by

$$P(t) = \sum_{e=0}^{\mathrm{inf}} |N_e| \, p^{-ne} t^e$$

This is very useful for finding the number of solutions congruent to 0 mod p^e. It is also very strongly related to the Igusa local Zeta function by

$$Z(t) = P(t) - t^{-1}\left(P(t) - 1\right)$$

It is know that the zeta function (and therefore the Poincaré series)is rational function with respect to t where

$$t = p^{-s}$$

This was show by resolution of singularities by Igusa in 1975.

5. Stationary Phase Formula

Earlier we defined the Igusa Local Zeta function of $f(x)$ to be

$$Z(s) = \int_{Z_p^n} |f(x)|^s \, dx$$

One of the ways to evaluate this integral is to determine in which anuali the function will return the same absolute value, and then use the measure of these sets to find the integral. These values were all based upon setting

$$\int_{Z_p^n} = 1$$

The stationary phase method is very similar process to suming over the anuali. It splits the integral up into three different parts so

$$Z(s) = \left(p^n - |N_1|\right)p^{-n} + \left(|N_1| - |S|\right)p^{-n}t\frac{1-p^{-1}}{1-p^{-1}t} + \sum_{a\in S}\int_{a+p\mathbb{Z}_p^n} |f(x)|^s \, dx$$

In this formula, the set N_1 is defined as

$$N_1 = \left\{x \in \mathbb{F}_p^n | f(x) \equiv 0 \ \mathrm{mod} p\right\}$$

whereas S is defined by

$$S = x\left\{\in N_1\right\}$$

$$f(x) \equiv 0 \text{ mod } p$$

$$\frac{\partial f}{\partial x_i}(x) \equiv 0 \text{ mod } p$$

The first part of SPF, $\left(p^n - |N_1|\right)p^{-n}$ represents all of the places in $\left(\mathbb{Z}_p - p\mathbb{Z}_p\right)^n$ which $|f(x)|$ has value of one. Since each one of these sets have measure p^{-n}, and there are $p^n - |N_1|$ of them, they contribute to the Zeta function

$$\left(p^n - |N_1|\right)p^{-n}$$

The next part of SPF comes from all of the points in $|N_1|$, but the first derivative does not vanish modulo p for all of the partial derivatives. These points are very similiar to finding the Zeta function for $f(x) = x$ because both parts have a derivate that can not equal zero. There are a total of $(|N_1| - |S|)$ cosets that each contribute

$$p^{-n}t\frac{1-p^{-1}}{1-p^{-1}t}$$

The last part of the formula contains all of the remaining points that were not integrated over. However this is often much easier than the original integral to compute. Using a substitution method, this new integral can usually be solved though SPF in a iterative method similar to $\int xcos(x)dx$.

6. IGUSA LOCAL ZETA FUNCTION FOR A PARTICULAR $f(x,y)$ CLASS USING SPF

For polynomials of the class $f(x,y) = x^4 + ax^2y^2 + y^4$ in which $a^2 - 4 \not\equiv 0 \bmod p$, the zeta function is defined as:

$$Z(s) = \int_{Z_p^n} |f(x)|^s \, dx$$

From the Stationary Phase Formula, we have [?]

$$Z(s) = \int_{Z_p^n} |f(x)|^s \, dx = (p^n - |N_1|)\, p^{-n} + (|N_1| - |S|)\, p^{-n}t\frac{1-p^{-1}}{1-p^{-1}t} + \int_S |f(x)|^s \, dx$$

where $t = p^{-s}$. The set N_1 is defined as

$$N_1 = \{(x,y)\,|f(x,y) \equiv 0 \bmod p\}$$

N_1 is difficult to compute because it depends on the prime number p, and also the possible values of a in $f(x,y)$. A much easier calculation is that for $|S|$. The set S consists of all $\{x,y\} \in N_1 \subset Z_p^2$ such that:

$$\begin{aligned} f(x,y) = x^4 + ax^2y^2 + y^4 &\equiv 0 \bmod p \\ \frac{\partial f}{\partial x} = 4x^3 + 2ay^2x &\equiv 0 \bmod p \\ \frac{\partial f}{\partial y} = 4y^3 + 2ax^2y &\equiv 0 \bmod p \end{aligned}$$

One of the singular points is $\{0,0\}$ since it satisifies all three equations. Since $x \equiv 0 \bmod p$ if and only if $y \equiv 0 \bmod p$, we can suppose for all other possible singular points, $x, y \not\equiv 0 \bmod p$. From this, the partial differential equations become

$$\begin{aligned}\frac{\partial f}{\partial x} &\equiv 2x^2 + ay^2 \equiv 0 \bmod p\\ \frac{\partial f}{\partial y} &\equiv 2y^2 + ax^2 \equiv 0 \bmod p\end{aligned}$$

Or simply

$$\begin{aligned}x^2 &\equiv \frac{-a}{2}y^2 \bmod p\\ y^2 &\equiv \frac{-a}{2}x^2 \bmod p\end{aligned}$$

Solving we find that

$$y^2 \equiv \frac{-a}{2}\left(\frac{-a}{2}y^2\right) \equiv \frac{a^2}{4}y^2 \bmod p$$

Cancelling out y^2, since $y \not\equiv 0$, we have

$$1 \equiv \frac{a^2}{4} \bmod p \Rightarrow a^2 - 4 \equiv 0 \bmod p$$

which contradicts our earlier assumption. Therefore the only singular point is $\{0, 0\}$ so then $|S| = 1$ Adjusting for our equation, we find

$$Z(s) = \left(p^2 - |N_1|\right)p^{-2} + \left(|N_1| - 1\right)p^{-2}t\frac{1 - p^{-1}}{1 - p^{-1}t} + \int_{(0+pZ_p)(0+pZ_p)} \left|x^4 + ax^2y^2 + y^4\right|^s dx$$

Using a change of variables such that $w = xp$ and $v = yp$ we find that the measure changes by $dxdy = p^{-2}dwdv$ Thus the zeta function becomes

$$Z(s) = \left(p^2 - |N_1|\right)p^{-2} + \left(|N_1| - 1\right)p^{-2}t\frac{1 - p^{-1}}{1 - p^{-1}t} + p^{-2}I$$

where

$$I = \int_{Z_p^2} \left|p^4x^4 + ap^4x^2y^2 + p^4y^4\right|^s dx = p^{-4s}\int_{Z_p^2} \left|x^4 + ax^2y^2 + y^4\right|^s dx$$

So we obtain the recursive relationship for $Z(s)$:

$$Z(s) = \left(p^2 - |N_1|\right)p^{-2} + \left(|N_1| - 1\right)p^{-2}t\frac{1 - p^{-1}}{1 - p^{-1}t} + p^{-2}t^4Z(s)$$

We can thus solve for $Z(s)$ as:

$$Z(s) = \frac{\left(1 - p^{-1}t\right)\left(p^2 - |N_1|\right)p^{-2} + \left(|N_1| - 1\right)p^{-2}t\left(1 - p^{-1}\right)}{\left(1 - p^{-2}t^4\right)\left(1 - p^{-1}t\right)}$$

7. Igusa Local Zeta function for particular $f(x,y,z)$ class using SPF

For polynomials of the class $f(x,y,z)=(x-y)^2+z$ the zeta function can be found by the stationary phase formula to be

$$\begin{aligned}Z(s) &= \int_{Z_p^3} |f(x,y,z)|^s\,dx \\ &= \left(p^3-|N_1|\right)p^{-3}+\left(|N_1|-|S|\right)p^{-3}t\frac{1-p^{-1}}{1-p^{-1}t}+\int_S |f(x,y,z)|^s\,dx\end{aligned}$$

Since the partial derivative

$$\frac{\partial f}{\partial z}=1 \text{ for all } \{x,y,z\}$$

there are no singular points for this polynomial. This implies that $|S|=0$ We easily see $|N_1|=p^2$ Hence the Zeta function becomes:

$$Z(s)=\left(p^3-p^2\right)p^{-3}+\left(p^2\right)p^{-3}t\frac{1-p^{-1}}{1-p^{-1}t}=\frac{1-p^{-1}}{1-p^{-1}t}$$

8. Newton Polyhedria Method

Besides SPF, there is a second method to compute the Zeta function. This is performed by using the function $f(x)$ to create a geometric picture, which then separtes all of the cosets of $\mathbb{Z}_p^n$ in which $|f(x)|^s$ is very similar.

Definition 8.1. The *support* ofa function $f=\sum_{w\in\mathbb{N}^n} a_w x_1^{w_1}...x_n^{w_n}$, a nonzero polynomial over $\mathbb{Z}_p^n$ is defined as:

$$\operatorname{supp}(f)=\{w\in\mathbb{N}^n | a_w\neq 0\}$$

From this we can define the Newton polyhedron

Definition 8.2. The *Newton polyhedron*, $\Gamma(f)$ of f is defined as the convex hull in $(\mathbb{R}^+)^n$ of the set:

$$\bigcup_{w\in\operatorname{supp}(f)} w+\left(\mathbb{R}^+\right)^n$$

One thing to note from this is that not all of the points in the supp(f) will define the shape of the Newton polyhedron. We have all of the proper faces of the Newton polyhedron, and the improper face of the entire polydedron, $\Gamma(f)$.

Definition 8.3. We define the minimum over the supp(f) as

$$m(a) = \mathrm{inf}_{x \in \Gamma(f)}\{a \cdot x\}$$

for $a \in (\mathbb{R}^{+})^{n}$

This is needed in order to define the first meet locus.

Definition 8.4. The *first meet locus of* a is

$$F(a) = \{x \in \Gamma(f) | a \cdot x = m(a)\}$$

where $a \cdot x$ denotes the usual scalar product.

The shape of the newton polygon with these definitions allow us to partition $(\mathbb{R}^{+})^{n}$ into cones using the equivalence relation:

$$a \sim a' \text{ if and only if } F(a) = F(a')$$

Definition 8.5. If τ is a face of the Newton Polyhedron, then the *cone associated with* τ is

$$\Delta_\tau = \{a \in (\mathbb{R}+)^{n} \, | F(a) = \tau\}$$

Once we have split up $(\mathbb{R}+)^{n}$, we can now use these cone partitions to integrate.

$$Z(s) = \sum_{\tau} \sum_{\mathbb{N}^3 \bigcap (k_1,..,k_n) \in \Delta_\tau} \int_{\left(p^{k_1}\mathbb{Z}_p^*\right) \times \ldots \times \left(p^{k_n}\mathbb{Z}_p^*\right)} |f(x)|^{s} \, dx$$

Using the substitutions

$$x_1 = p^{k_1} u_1$$
$$x_2 = p^{k_2} u_2$$
$$\ldots$$
$$x_n = p^{k_n} u_n$$

The Zeta function now becomes

$$Z(s) = \sum_{\tau} \sum_{\mathbb{N}^3 \bigcap (k_1,..,k_n) \in \Delta_\tau} p^{-k_1 - k_2 - \ldots - k_n} \int_{\left(\mathbb{Z}_p^*\right)^{n}} \left| f(p^{k_1}u_1, p^{k_2}u_2, \ldots, p^{k_n}u_n)\right|^{s} dx$$

We must associate parts of f with faces of the the Newton polyhedron.

Definition 8.6. For every face τ of the Newton polyhedron, the associated part of f with that face of the Newton polyhedron is

$$f_\tau(x) = \sum_{w \in \tau} a_w x^{w}$$

Note that $f_{\Gamma(f)} = f$, so the entire face of the the polygon is included in this definition.

Definition 8.7. A function f is *non-degenerate* over $\mathbb{F}_p$ with respect to the faces of its Newton polyhedron if for every face τ of $\Gamma(f)$, the set of congruences

$$\begin{aligned} f_\tau(x) &= 0 \bmod p \\ \frac{\partial f_\tau}{\partial x_i}(x) &= 0 \bmod p \text{ for } i = 1, .., n \end{aligned}$$

has no solution in $\left(\mathbb{Z}_p^*\right)$

In the special case of non-degeneracy, the Zeta function simplifies into two different parts over each face of the polynomial.

Lemma 8.8. *Let f be a function that is non-degenerate with respect to each face of its Newton polyhedron. Then we have that the zeta function becomes*

$$Z(s) = \sum_\tau L_\tau S_{\Delta\tau}$$

for each face τ in the Newton polyhedron where

$$L_\tau = p^{-n}\left((p-1)^n - p\,|N_\tau|\,\frac{p^s-1}{p^{s+1}-1}\right)$$

In this, we define the set

$$N_\tau = \left\{x \in \left(\mathbb{Z}_p^*\right)^n | f_\tau(x) \equiv 0\right\}$$

Also we have

$$S_{\Delta\tau} = \sum_{k \in \mathbb{N}^n \bigcap \Delta_\tau} p^{-\sigma(k) - m(k)s}$$

Where we define $\sigma(a) = \sum_{i=1}^n a_i$

The proof for this is give in Kathleen Hoornaert's thesis [**?**]

A better working definition for $S_{\Delta\tau}$ in the case when the cone Δ_τ is spanned by linearly independent vectors $a_1, ..., a_r$ is

$$S_{\Delta\tau} = \sum_h \frac{p^{-\sigma(h) - m(h)s}}{\left(p^{\sigma(a_1)+m(a_1)s} - 1\right) .... \left(p^{\sigma(a_r)+m(a_r)s} - 1\right)}$$

9. Using Polygusa for degenerate polynomials with respect to the Newton Polyhedra

If the function $f\left(x_1,...,x_n\right)$ is degenerate, the system of equations:

$$f\left(x_1,...,x_n\right) \equiv 0 \text{ mod } p$$
$$\frac{\partial f}{\partial x_i}\left(x_1,...,x_n\right) \equiv 0 \text{ mod } p$$

has a nonzero solution $(a_1,...,a_n) \in \left(\mathbb{F}_p^*\right)^n$. The formula given for the Igusa local Zeta function uses the Newton polyhedra of f to express

$$Z(s) = \sum_{\tau \in \Gamma(f)} L_\tau S_{\Delta\tau}$$

where L_τ depends on f_τ being nondegenerate. Otherwise the formula is not correct. Kathleen Hoorneart along with Davy Loots wrote a program called Polygusa for computing the Igusa local Zeta function. This Maple program relies on previous code written by Fukuda. Because the program uitalizes the Newton Polygon method, it is not acceptable for finding the Zeta function of degenerate polynomials. For example in Polygusa, an error message is displayed when it is asked to compute the Zeta function of $f = (x-y)^2 + z$.

To avoid the check for degeneracy, the command **nocheck** is added. An example of this is the code:

```
> Zetafunction((x-y) ^ 2+z,[x,y,z],igusa, nocheck))
```

The output from this command gives a Zeta function for the polynomial, based purely on what the calculated $L_\tau S_{\Delta\tau}$ values for each cone . However because one of the faces is degenerate, the resulting Zeta function is incorrect.

In order to determine how each face of the polygon contributes to the total zeta function, the command **faceinfo** is further added. Combining both commands together we have

```
> Zetafunction((x-y) ^ 2+z,[x,y,z],igusa, nocheck, faceinfo))
```

This allows for each face's contribution to the Zeta function to be found. Since the Zeta function splits into

$$\begin{aligned} Z(s) = & \sum_{\tau \text{ non-degenerate}} L_\tau S_{\Delta\tau} \\ & + \sum_{\tau \text{ degenerate}} \sum_{\mathbb{N}^3 \bigcap (k_1,..,k_n) \in \Delta_\tau} \int_{\left(p^{k_1}\mathbb{Z}_p^*\right) \times ... \times \left(p^{k_n}\mathbb{Z}_p^*\right)} |f(x)|^s \, dx \end{aligned}$$

by finding the correct contribution from each degenerate face and using a program like Polygusa to calculate the contribution from each non-degenerate face allows the total Zeta function to be found.

10. KATHLEEN'S CONJECTURES

In her 2002 thesis, Kathleen Hoonaert makes two conjecture concerning the poles of a zeta function [?]:

Conjecture 10.1. Let $f\left(x_1,..,x_n\right)$ be a polynomial over $\mathbb{Z}_p$ with $f(0)=0$ and p a prime number. Suppose that f is non-degenerate over $\mathbb{C}$ with respect to all the faces of its Newton polyhedron $\Gamma(f)$ and that the origin is a singular point of f.

If $t_0<1$ then $-1/t_0$ is a pole of $Z_f(s)$ for $p \gg 0$.

Conjecture 10.2. Let $f\left(x_1,..,x_n\right)$ be a polynomial over $\mathbb{Z}_p$ with $f(0)=0$ and p a prime number. Suppose that f is non-degenerate over $\mathbb{F}_p$ with respect to all the faces of its Newton polyhedron $\Gamma(f)$ and that the origin is a singular point of f. Suppose that τ_0 is not unstable over $\mathbb{F}_p$ with respect to f. If $t_0<1$, then $-1/t_0$ is a pole of $Z_f(s,\chi)$ of order κ for at least one character χ.

Remark 10.3. It is only necessary to look at $f\left(x_1,...,x_n\right)$ for $n>2$ in these two conjectures since in the $n=2$ cases, there does not exist any function $f(x,y)$ that satisfy her assumptions

Proof. Assume there exists a function $f(x,y)$ that satisify's all of Kathleen's assumptions. In order for $t_0<1$, there must be a face of the polygon that intersects the line $y=x$ for some $x<1$. Either the intersection occurs at a face of degree zero or one.

CASE 1: If the intersection occurs at a face of degree 0, the only possible face would be $(0,0)$. This would imply a constant term in $f(x,y)$. But this causes a contridiction since we assumed $f(0,0)=0$.

CASE 2: The interesction occurs at a face of degree one. We will show that this line is defined by $(0,y_1)$ and $(x_2,0)$ and that either $y_1=1$ or $x_2=1$.

For any face between the support points (x_1,y_1) and (x_2,y_2) the line forming the face between these two points is defined by the set

$$\{(x_1,y_1)t+(1-t)((x_2,y_2)\}$$

for $0\leq t\leq 1$. When this face intersects $y=x$ at (t_0,t_0) we find

$$(t_0,t_0)=(x_1,y_1)t+(1-t)((x_2,y_2)$$

This equation is equivalent to both

$$t_0=x_1t+(1-t)x_2$$
$$t_0=y_1+(1-t)y_2$$

Consider that $x_1, x_2 \geq 1$. Assume without loss of generality, $x_1 \leq x_2$. Thus

$$t_0 = (1-t)x_1 + tx_2 \geq (1-t)x_1 + tx_1 = x_1 = \text{Min}\,\{x_1, x_2\} \geq 1$$

But t_0 is assumed to be less than 1. So either $x_1 < 1$ or $x_2 < 1$. By a similar argument either $y_1 < 1$ or $y_2 < 1$. Without a loss of generality, let $x_1 = 0$. Therefore the point $(0, y_1)$ is in the support of f. If $y_1 = 0$, this causes a constant term to appear in f, making $f(0,0) \neq 0$ and violating the assumption that $f(0,0) = 0$. Therefore $y_2 = 0$ Thus the above two equations reduce down to

$$\begin{aligned} t_0 &= tx_2 \\ t_0 &= (1-t)y_1 \end{aligned}$$

Solving for t, we find

$$t_0 = \left(1 - \frac{t_0}{x_2}\right) y_1$$

Since y_1 is positive, $0 < t_0 < 1$, and assuming $x_2 > 1$, we find

$$t_0 = y_1 \left(1 - \frac{t_0}{x_2}\right) = y_1 \left(\frac{x_2 - t_0}{x_2}\right) \geq y_1 \left(\frac{x_2 - 1}{x_2}\right) \geq y_1(1/2)$$

This implies that either $x_2 = 1$ or $y_1 = 1$. Without a loss of generality, this makes $f = a_1 y^{y_1} + a_2 x + yg(x,y) + x^2 q(x)$ where $g(x,y)$, $q(x)$ are other polynomials over $\mathbb{Z}_p$. When the partial derivative with respect to x is taken, considering only $p \gg 0$,

$$\frac{\partial f}{\partial x}(x,y) \equiv a_2 + y\left(\frac{\partial g(x,y)}{\partial x}\right) + x^2\left(\frac{\partial q(x)}{\partial x}\right) + 2x\,(q(x))$$

It follows that

$$\frac{\partial f}{\partial x}(0,0) = a_2 + 0\left(\frac{\partial g(0,0)}{\partial x}\right) + 0\left(\frac{\partial q(0)}{\partial x}\right) + 0(q(0)) \equiv a_2 \bmod p$$

As long as $a_2 \not\equiv p$, the origin is not a singular point causing a contridiction of the assumptions. However if $a_2 \equiv p$, then $f(x,y)$ is degenerate with respect to its Newton Polygon, also contridicting an assumption. Since all of the cases are exhausted, there is no possible such $f(x,y)$ that satisifies Kathleen's assumptions. [?] □

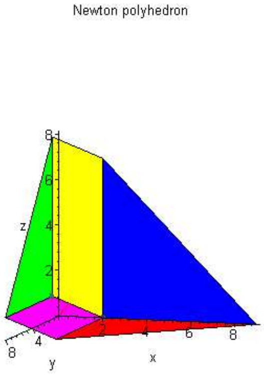
11. IGUSA LOCAL ZETA FUNCTION OF $f(x,y,z) = x^2 + yz + z^2xy$ USING THE NEWTON POLYHON METHOD

This example is very close to that given in Kathleen Hoonaert's thesis [?]. Upon taking the zeta function of this we find using the Newton Polyhedra method and the program Polygusa

$$Z(s) = \frac{(p-1)\left(p^{3s}p^5 + p^{3s}p^3 - p^{2s}p^3 - p^{2s}p^2 - p^s + 1\right)}{p^2\left(p^sp-1\right)\left(p^{2s}p^3-1\right)}$$

[?]

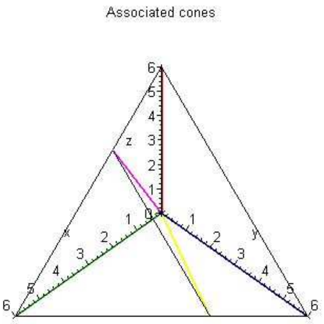
For the non-trivial f_τ, $|N_\tau|$ were determined through experimental methods using a program written in gp pari. For $f_\tau = x^2 + yz$, the number of nonzero solutions was $(p-1)^2$. The number of nonzero solutions of $f(x,y,z)$ were $(p-1)(p-2)$. All of the other f_τ, the number of nonzero solutions were $p^{-3}(p-1)^3$. The polyhedra is defined by the convex hull of the points $(2,0,0)$, $(0,1,1)$ and $(1,1,2)$ in the support of $f(x,y,z)$. For this example, $t_0 = 2/3$ since the plot of the line $y = x = z$ intersects the newton polyhedra at the point $(2/3, 2/3, 2/3)$ on a face of dimention one.



This face is defined by the the set of points

$$(x,y,z) = \{(0,1,1) + t(1,0,1) | 0 \le t \le 1\}$$

When splitting the polyhedron into faces, the Newton polyhedron also partitions $\mathbb{R}^{+3}$ into cones associated with each face of the polyhedron.



Remark 11.1. To better understand why the root $t_0 = 2/3$ only has multiciplicity one, we examined all of the faces of the newton polygon that could contribute the root with multiciplicity two. This corresponed to the cones defined by the vectors $(1,2,0)$ and $(1,0,2)$ and any other vectors. Therefore only three possible faces could cause the root $t_0 = 2/3$ to have multiplicity two. One of them is one dimensional, and the other two are zero dimensional. Let κ be the set that contains these three faces. We we look at the contribution of the zeta function from this set we find

$$\sum_{\tau\in\kappa} L_\tau S_{\Delta_\tau} =$$

$$\frac{(p-1)^2\left(p^2p^s-2pp^s+1\right)\left(p^3p^{2s}+1\right)}{p^3\left(p^3p^{2s}-1\right)^2\left(pp^s-1\right)}+$$

$$\frac{(p-1)^2\left(p^3p^{2s}+2p^2p^s+1\right)}{p^3\left(p^3p^{2s}-1\right)^2}+$$

$$\frac{(p-1)\left(2p^4p^{2s}-p^3p^{2s}-1\right)}{p^3\left(p^3p^{2s}-1\right)^2}$$

$$=\frac{\left(p^3p^s-1\right)}{p^3\left(p^3p^{2s}-1\right)\left(pp^s-1\right)}$$

Thus the second multiplicity of the root disappears from this part of the zeta function. Since only these $L_\tau S_{\Delta\tau}$ contribute this root to the second order, the rest of the $L_\tau S_{\Delta\tau}$ can only cause t_0 to have multiplicity one which goes against what we earlier suspected.

This example is very similar to one in Kathleen's paper, although this polygon has a much different shape. I would like to examine if the nontrivial f_τ causes the root to lose mulitplicity in the faces that contribute the root of highter order. More specifically, I would like to see how much of the loss of the root is due to the geometry of the polygon, and the S_{Δ_τ}. In some other examples, we have found that it is not enough to look at only the $L_\tau S_{\Delta_\tau}$ that contribute the higher order. Therefore I would like to examine the class of functions to which the the higher order of the root disappears in the faces that contribute the higher order root.

12. Conclusion

We have examined the Igusa local Zeta function and some methods used to peform the calculation. Using the Newton polygon method, we have looked into a geometric interpretation of where the roots of the zeta function come from. In the future, we would like to continue on this path of exploration, especially into the cases where the roots of the zeta function occur with multiplicity less than expected.

13. Acknowledgement

The author would like to thank Professor Margaret Robinson for her help and support in this project. They are also indebted to Jo Miles for her contributions and collaboration, especially in the proof of remark 10.3. We thank the Mount Holyoke Research Experience for Undergraduates 2005 program for the opportunity to perform this work as well as the other participants for their encouragement and collaborations. Finally we thank the NSF for their sponsership.

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