

# LOCAL DERIVATIVES AND BERNSTEIN POLYNOMIALS

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ABSTRACT. We introduce the local derivatives of a Weyl algebra and prove a theorem of I. N. Bernstein concerning the existence of certain polynomials relating to the action of local derivatives.

## OVERVIEW

The organization of this paper is as follows. In the first section, we introduce  $D$ -modules and the notion of local derivatives of polynomial rings. We also state Bernstein's theorem in the language of these local actions. The second section contains a short review of results on filtrations to be used in our proof. The third section focuses on  $(d, e)$ -filtrations originally studied by Bernstein in [1]. Finally, in the last section we complete the rest of the proof using  $(d, e)$ -filtrations.

## 1. LOCAL ACTIONS OF THE WEYL ALGEBRA

Let  $K$  be a field of characteristic zero. Denote  $K[X] := K[x_1, \dots, x_n]$ .

**Definition 1.1.** The  $n$ th Weyl algebra  $D_n$  is the  $K$ -subalgebra of  $\text{End}_K(K[X])$  generated by the linear operators  $x_i$  and  $\partial/\partial x_i$  where  $x_i$  is multiplication by a variable and  $\partial/\partial x_i$  is a formal partial derivative.

The generators of  $D_n$  satisfy the following relations:

- $x_i x_j = x_j x_i$ ,
- $\partial/\partial x_i (\partial/\partial x_j) = \partial/\partial x_j (\partial/\partial x_i)$ ,
- $(\partial/\partial x_j) x_i = x_i (\partial/\partial x_j) + \delta_{ij}$ .

These are known as the Heisenberg commutation relations; they give a complete characterization of  $D_n$  in the following sense.

Let  $V$  be a  $2n$ -dimensional  $K$ -vector space with basis  $\{\varepsilon_1, \dots, \varepsilon_n, \eta_1, \dots, \eta_n\}$ . Consider the ideal  $\mathcal{I}(V)$  of the tensor algebra  $\mathcal{T}(V)$  generated by elements of the form

$$\varepsilon_i \otimes \varepsilon_j - \varepsilon_j \otimes \varepsilon_i, \quad \eta_i \otimes \eta_j - \eta_j \otimes \eta_i, \quad \eta_j \otimes \varepsilon_i - \varepsilon_i \otimes \eta_j - \delta_{ij}.$$

**Proposition 1.2.** *The  $K$ -linear map  $\phi : \mathcal{T}(V)/\mathcal{I}(V) \rightarrow D_n$  defined by  $\phi(\varepsilon_i) = x_i$  and  $\phi(\eta_i) = \partial/\partial x_i$  is an isomorphism.*

*Proof.* The surjectivity of  $\phi$  is clear. Let  $w$  be such that  $\phi(w) = 0$ . Write

$$\phi(w)x_j = \sum c_{ij} x_i (\partial/\partial x)_j, \quad x_i = x_1^{i_1} \dots x_n^{i_n}, \quad \dots$$

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Suppose that  $w \neq 0$ ; then there is a nonzero  $c_{ij}$ . Choose  $c_{ij}$  such that  $|j| = \sum j_k$  is minimal. We have

$$\phi(w)x_j = \sum_i j!c_{ij}x_i, \quad j! = \prod_{i=1}^n j_k!.$$

The product  $j!$  is non-zero since  $\text{char}(K) \neq 0$ , so  $\phi(w)x_j$  must be non-zero. This contradicts our choice of  $w$ .  $\square$

Let  $L = K(s)$  be the field of rational functions in one variable over  $K$ . Let the Weyl algebra  $D_n$  be defined over  $L$ . The field  $L(X)$  is a  $D_n$ -module as described above. We now introduce the notion of a local derivative.

**Definition 1.3.** Let  $f$  be a non-zero element of  $K[X]$ . The  $i$ th partial derivative localized at  $f$  is the operator  $\eta_i$  is defined for  $g \in L(X)$  by

$$\eta_i \cdot_f g = \frac{\partial g}{\partial x_i} + \frac{sg}{f} \left( \frac{\partial f}{\partial x_i} \right).$$

In the case that  $f$  is a constant, this reduces to the standard partial derivative. We can thus generalize the definition of the Weyl algebra to local Weyl algebras which are generated by variables and local derivatives at possibly non-constant  $f$ .

However, it is preferable to think of  $D_n$  as a quotient of the tensor algebra with multiple local actions on  $L(X)$  as a module; this allows us to consider all localizations simultaneously within a single object. We use the notation  $P \cdot_f \varphi$  to denote the operator  $P$  localized at  $f$  acting on  $\varphi$ . If we are concerned only with a fixed local action, we write  $D_n(L, f)$  to indicate this restriction and the base field.

Let  $L[X]_f$  to be the localization of  $L[X]$  at  $S = \{f^e : e \in \mathbf{N}\}$ ; this is a module over  $D_n(L, f)$ . As it turns out, this is precisely the module generated by the  $f$ -local action on  $L[X]$ . We first proceed to the following theorem of Bernstein.

**Theorem 1.4** (Bernstein). *There exists  $P \in D_n(L)$  such that  $P \cdot_f f = 1$ .*

This result implies that there exists a polynomial  $b(s)$  such that

$$P \cdot_f f = b(s)$$

where  $P = b(s)P_0$  is an element of  $D_n(L)$  with coefficients in  $K[s]$ . The polynomial  $b(s)$  is not unique, but the set of all such  $b(s)$  forms an ideal  $B_f \subset K[s]$ .

**Definition 1.5.** The Bernstein polynomial  $b_f(s)$  is the monic generator of  $B_f$ .

In addition, we have the following formal relation between local actions at  $f$  and the constant local action:

$$P \cdot_c f^{s+1} = (P \cdot_f f)f^s = b_f(s)f^s.$$

To address a possible point of confusion,  $f^s$  lies not in  $L[X]_f$ , but in a larger ring of differentiable functions; the above equality is purely formal, and becomes useful in the complex analytic setting.

**Example 1.6.** Let  $f(x, y) = x^3 + y^3$ . Let

$$P = x \frac{\partial^4}{\partial x^4} + 3x \frac{\partial^4}{\partial y^3 \partial x} + 6 \frac{\partial^3}{\partial x^3} + 6 \frac{\partial^3}{\partial y^3} + 3y \frac{\partial^4}{\partial x^3 \partial y} + y \frac{\partial^4}{\partial y^4}.$$

Then  $P \cdot_c f^{s+1} = 9(s+1)^2(3s+2)(3s+4)f^s$  and the Bernstein polynomial is

$$b_f(s) = (s+1)^2(s+2/3)(s+4/3).$$

We will not prove 1.4 directly, but instead formulate an equivalent condition and prove that.

**Proposition 1.7.** *Theorem 1.4 holds if and only if  $L[X]_f$  is finitely generated as a  $D_n(L, f)$ -module.*

For the proof, we first establish a lemma.

**Lemma 1.8.** *Let  $\varphi(s) \in L[X]_f$  and  $P(s) \in D_n(L, f)$ . Then*

$$P(s) \cdot (f^{-r}\varphi(s)) = f^{-r}(P(s+r) \cdot \varphi(s+r))|_{s \mapsto s-r}$$

where  $P(s+r)$  is the image of  $P(s)$  under the endomorphism  $s \mapsto s+r$  and the notation  $\phi(s)|_{s \mapsto s-r}$  is defined to be  $\phi(s-r)$ .

*Proof of lemma.* The statement holds for  $P = x_i$  and  $P = \partial/\partial x_i$ . Observe that if the formula holds for  $P_1, P_2$  then it holds for  $c_1P_1 + c_2P_2$ . We now show that it holds for  $P_1P_2$ :

$$\begin{aligned} P_1(s)P_2(s) \cdot f^{-r}\varphi(s) &= P_1(s) \cdot f^{-r}(P_2(s+r) \cdot \varphi(s+r))|_{s \mapsto s-r} \\ &= f^{-r}\{(P_1(s+r) \cdot [P_2(s+r) \cdot \varphi(s+r)])|_{s \mapsto s-r}\} \\ &= f^{-r}(P_3(s+r) \cdot \varphi(s+r))|_{s \mapsto s-r} \end{aligned}$$

□

*Proof of 1.7.* Let  $r$  be a non-negative integer. We begin with some observations:

$$D \cdot f^{-r+1} = Df \cdot f^{-r} \subset D \cdot f^{-r}, \quad L[X]f^{-r} = L[X] \cdot f^{-r} \subset D \cdot f^{-r}.$$

Consequently, we have that  $D \cdot 1 \subset D \cdot f^{-1} \subset D \cdot f^{-2} \subset \dots$  and therefore

$$\bigcup_{r \geq 0} D \cdot f^{-r} = L[X]_f.$$

Suppose that  $L[X]_f$  is generated by the finite set  $S$ . We can then choose  $r$  such that  $S \subset D \cdot f^{-r+1}$ , thus giving  $L[X]_f = D \cdot f^{-r+1}$ . Then  $f^{-r+1} = P(s) \cdot f^{-r+1}$  for some  $P(s)$ . From lemma 1.8 above, we conclude that  $P(s+r) \cdot f = 1$ .

Suppose now that  $P \cdot f = 1$  for some  $P(s)$ . By lemma 1.8 we have

$$P(s-1) \cdot 1 = f^{-1}, \dots, P(s-r) \cdot f^{-r+1} = f^{-r}$$

for all non-negative  $r$  and thus  $L[X]_f = D \cdot 1$ . □

From this, we can derive our earlier claim about the minimality of  $L[X]_f$  as a submodule containing  $L[X]$  from Bernstein's theorem.

**Corollary 1.9.** *The submodule  $L[X]_f$  is the minimal  $D_n(L, f)$ -submodule of  $L(X)$  containing the subring  $L[X]$ .*

*Proof.* If theorem 1.4 holds, then by lemma 1.8 we have  $P(s-1) \cdot 1 = f^{-1}$ . □

The proof of the finite generation of  $L[X]_f$  requires some machinery from commutative algebra, which we now review.

## 2. BACKGROUND ON FILTRATIONS

Let  $K$  be an arbitrary field,  $A$  a  $K$ -algebra with unit, and  $M$  an  $A$ -module. Our strategy for determining whether  $M$  is finitely generated will be to take filtrations and investigate the induced graded  $G(A)$ -module  $G(M)$ .

**Definition 2.1.** A filtration of a  $K$ -algebra  $A$  is a system  $\{A_i\}$  of subspaces such that

- $A_0 \subset A_1 \subset \cdots$ ,
- $\bigcup_i A_i = A$ ,
- $A_i A_j \subset A_{i+j}$  for all  $i, j$ .

A filtered  $K$ -algebra is a  $K$ -algebra with a filtration.

**Definition 2.2.** Let  $A$  be a filtered  $K$ -algebra. A filtration of an  $A$ -module  $M$  is a system  $\{M_i\}$  of submodules such that

- $M_0 \subset M_1 \subset \cdots$ ,
- $\bigcup_i M_i = M$ ,
- $A_i M_j \subset M_{i+j}$  for all  $i, j$ .

A filtered  $A$ -module is an  $A$ -module with a filtration.

Two filtrations  $M_i, M'_i$  of  $M$  are said to be *equivalent* if there exist integers  $r, s$  such that for all  $i$  we have

$$M_i \subset M'_{i+r}, \quad M'_i \subset M_{i+s}.$$

This is clearly an equivalence relation.

**Definition 2.3.** The associated graded algebra  $G(A)$  of a filtered  $K$ -algebra  $A$  is the  $K$ -algebra  $\bigoplus_i G_i(A)$  where  $G_i(A) = A_i/A_{i-1}$ .

**Definition 2.4.** The associated graded module  $G(M)$  of a filtered  $A$ -module  $M$  is the  $A$ -module  $\bigoplus_i G_i(M)$  where  $G_i(M) = M_i/M_{i-1}$ .

*Remark 2.1.* The associated graded module functor from the category of filtered  $A$ -modules to the category of  $G(A)$ -modules is exact.

A filtration of  $M$  is said to be *standard* if its associated graded module  $G(M)$  is finitely generated as a  $G(A)$ -module.

**Proposition 2.5.** *If  $M$  has a standard filtration, then  $M$  is finitely generated.*

*Proof.* By definition,  $G(M)$  has a finite basis  $\{\beta_1, \dots, \beta_k\}$  over  $G(A)$ . Since each  $\beta_i$  is a finite sum of homogeneous elements, we can assume without loss of generality that  $\beta_i \in G_{r_i}(M)$  for some  $r_i \in \mathbf{N}$ . Let  $\alpha_i \in M_{r_i}$  be a lift of  $\beta_i$ . Define

$$F_r(M) = \sum_{i=1}^k A_{r-r_i} \alpha_i.$$

If  $F_r(M) = M_r$  for all  $r$ , then  $M = \sum A \alpha_i$ . It is clear that  $F_r(M) \subset M_r$ . The converse holds for  $r = -1$ . We induct on  $r$ . Let  $x \in M_r$  and denote its image in  $G_r(M)$  by  $y$ ; then  $y = \sum b_i \beta_i$  where  $b_i$  is in  $G_{r-r_i}(A)$  for all  $i$ . Let  $a_i$  be a lift of  $b_i$  in  $M_{r-r_i}$ ; then  $x - \sum a_i \alpha_i$  is in  $M_{r-1}$ . Therefore  $x$  is in

$$M_{r-1} + F_r(M) = F_{r-1}(M) + F_r(M) = F_r(M)$$

and we are done by induction. □

In fact, the converse and more hold under mild conditions.

**Proposition 2.6.** *Let  $A$  be a  $K$ -algebra and  $M$  be an  $A$ -module. If  $M$  is finitely generated, then  $M$  has a unique standard filtration up to equivalence.*

*Proof.* First we prove the existence of a standard filtration. Let  $M$  be generated by  $\{\beta_1, \dots, \beta_k\}$ . Define the surjective  $A$ -homomorphism

$$\phi : A^k \longrightarrow M, \quad (a_1, \dots, a_k) \mapsto a_1\beta_1 + \dots + a_k\beta_k.$$

Denote the kernel by  $N$ . Setting the filtrations

$$(A^k)_i = (A_i)^k, \quad M_i = A_i\beta_1 + \dots + A_i\beta_k, \quad N_i = (A_i)^k \cap N,$$

we get the exact sequence of filtered  $A$ -modules

$$0 \longrightarrow N \longrightarrow A^k \longrightarrow M \longrightarrow 0.$$

Since the associated graded module functor is exact, we have a surjective map  $G(A^k) \cong G(A)^k \longrightarrow G(M)$ . Hence,  $G(M)$  is finitely generated over  $G(A)$  and the filtration  $M_i$  is standard.

It remains to show uniqueness. Let  $M'_i$  be a standard filtration of  $M$ . As in the proof of 2.5, find a finite set of elements  $\alpha_i \in M'_{r_i}$  such that for all  $r$ ,

$$M'_r = \sum_{i=1}^k A_{r-r_i}\alpha_i$$

and thus  $M' = \sum_{i=1}^k A\alpha_i$ . Writing one basis in terms of the other, we have

$$\alpha_i = \sum_{j=1}^k b_{ij}\beta_j, \quad \beta_i = \sum_{j=1}^l a_{ij}\alpha_j.$$

Since the set  $S = \{a_{ij}, b_{ij}\}$  is finite, there is an  $r_0$  such that  $S \subset A_{r_0}$ . Let

$$s_0 = \max\{r_0 + r_1, \dots, r_0 + r_l\}.$$

We then have

- $M_r = \sum_{i=1}^l A_r\beta_i \subset \sum_{j=1}^k A_{r+r_0}\alpha_j \subset M'_{r+s_0}$ ,
- $M'_r = \sum_{j=1}^k A_{r-r_j}\alpha_j \subset \sum_{i=1}^l A_{r+r_0}\beta_i = M_{r+r_0}$ .

Thus,  $M_i$  and  $M'_i$  are equivalent. □

**Definition 2.7.** A short exact sequence of  $A$ -modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is short exact in the category of filtered  $A$ -modules if  $M', M, M''$  are filtered and for all  $i$ , there is an induced short exact sequence

$$0 \longrightarrow M'_i \longrightarrow M_i \longrightarrow M''_i \longrightarrow 0.$$

3.  $(d, e)$ -FILTRATIONS

We focus on a certain class of filtrations, introduced by Bernstein in [1], where the dimension grows at an order described by a polynomial of specified degree and coefficient.

**Definition 3.1.** Let  $K$  be a field,  $A$  be a filtered  $K$ -algebra, and  $M$  be a filtered  $A$ -module. If there exist integers  $e, d$  such that

$$\lim_{i \rightarrow \infty} \frac{\dim_K(M_i)}{i^d} = \frac{e}{d!}$$

then  $\{M_i\}$  is said to be a  $(d, e)$ -filtration.

*Remark 3.1.* Consider the exact sequence of filtered  $A$ -modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

such that  $M', M''$  are of type  $(d', e'), (d'', e'')$  respectively. Then  $M$  is of type  $(d, e)$  where  $d = \max(d', d'')$  and  $e = e', e' + e'', e''$  depending on  $d' > d'', d' = d'', d' < d''$ .

We now consider several examples, the last of which will come into play later.

**Example 3.2.** Let  $L = K$ ,  $D_n(L, f) = A$ , and  $L[X]_f = M$ . Define a filtration on  $D_n(L, f)$  by

$$A_i = \{P \in D_n(L, f) : \deg(P) \leq i\}$$

where  $\deg(P)$  is measured by regarding  $x_j$  and  $(\partial/\partial x_j)_f$  as variables of a polynomial ring. Define a filtration on  $L[X]_f$  by

$$M_i = \{f^{-i}g \in f^{-i}L[X] : \deg(g) \leq (\deg(f) + 1)i\}.$$

It is easily checked that this is a  $(n, (\deg(f) + 1)^n)$ -filtration.

**Example 3.3.** Let  $K$  be a field and  $A = M = K[x_1, \dots, x_m]$ . Define for all  $i$

$$M_i = \{f \in M : \deg(f) \leq i\}.$$

It then follows that

$$\dim_K(M_r) = (m + r)! / (m!r!)$$

and so  $\{M_i\}$  is a  $(m, 1)$ -filtration.

**Example 3.4.** Let  $A$  be a filtered  $K$ -algebra such that  $G(A) \cong K[x_1, \dots, x_m]$  for some  $m > 0$ . Let  $M$  be a non-zero finitely generated  $A$ -module with a standard filtration  $\{M_i\}$ . Then  $\{M_i\}$  is a  $(d, e)$ -filtration with  $d \geq m$ .

In fact,  $d$  and  $e$  in this last example are independent of the choice of filtration by 2.6. This allows us to make the following definition.

**Definition 3.5.** Under the conditions in 3.4, define  $d(M)$  and  $e(M)$  to be  $d$  and  $e$  respectively.

## 4. THE PROOF OF BERNSTEIN'S THEOREM

The following is Bernstein's key result on  $(d, e)$ -filtrations, from which all of our desired theorems follow.

**Theorem 4.1.** *We give two statements; the second assumes the first.*

- (1) *If  $M$  is a  $D_n(L, f)$ -module with a  $(d, e)$ -filtration, then  $d \geq n$ .*
- (2) *If  $d = n$ , then any ascending chain of submodules has length at most  $e$ .*

**Corollary 4.2.**  $L[X]_f$  is finitely generated as a  $D_n(L, f)$ -module.

**Lemma 4.3.** Let  $A$  be a filtered  $K$ -algebra with induced  $G(A)$  isomorphic to some polynomial ring. If  $M$  is a finitely generated  $A$ -module, then any  $A$ -submodule of  $M$  is finitely generated.

*Proof.* This is a simple case of the Hilbert Basis Theorem.  $\square$

*Remark 4.1.* We recall some facts about module actions under ring automorphisms, as well as establish notation for the next lemma.

**Lemma 4.4.** Let  $K$  be an uncountable algebraically closed field and  $M$  be a non-zero  $K[t]$ -module such that  $\dim_K(M)$  is countable. Let  $\phi : K[t] \rightarrow \text{End}_K(M)$  be the homomorphism induced by the module action. Then there exists  $\alpha \in K$  such that  $\phi(t - \alpha)$  is non-invertible.

*Proof.* Suppose that  $\phi(t - \alpha)$  is a unit for all  $\alpha \in K$ . Let  $f(t)$  be a non-zero polynomial in  $K[t]$ ;  $f$  can be written as  $f = c \prod (t - \alpha_i)$  since  $K$  is algebraically closed. As a product of units,  $\phi(f)$  is a unit. Hence, we can define a  $K(t)$ -module structure on  $M$  and expand the domain of  $\phi$  to  $K(t)$ . By hypothesis,  $M$  has a non-zero element  $m$ . Consider the map  $\psi : K(t) \rightarrow M$  by  $\beta \mapsto \phi(\beta)m$ . Observe that  $S = \{(t - \alpha)^{-1} : \alpha \in K\}$  is linearly independent over  $K$ . Since  $\dim_K(M)$  is countable and  $K$  is not,  $\psi$  is not injective. Therefore  $\phi(\beta)m = 0$  for some  $\beta \neq 0$ . However,  $\phi(\beta)$  is a unit and so  $m = 0$ , contradicting our choice of  $m$ .  $\square$

**Lemma 4.5.** Let  $\sigma, \tau \in \text{End}_K(D_n(K))$  be the maps such that

$$\begin{aligned} \sigma : \quad x_n &\mapsto x_n + \alpha, & \partial/\partial x_n &\mapsto \partial/\partial x_n + \beta, \\ \tau : \quad x_n &\mapsto -\partial/\partial x_n + \beta, & \partial/\partial x_n &\mapsto x_n + \alpha, \end{aligned}$$

where  $\alpha, \beta \in K$  and preserving the identity otherwise; then  $\sigma, \tau \in \text{Aut}_K(D_n(K))$ .

*Proof.* One can check that the Heisenberg relations hold under transformation.  $\square$

*Proof of 4.1 part 1.* Let  $M$  be a  $D_n(K, f)$ -module with  $(d, e)$ -filtration. The case  $M = 0$  is trivial. Assuming  $M \neq 0$ , there exists a finitely generated non-zero submodule  $M'$  of  $M$ ; therefore  $d(M') \leq d$ . We can therefore assume without loss of generality that  $M$  is finitely generated.

Since  $\dim_K(D_n)$  is countable and  $M$  is finitely generated,  $\dim_K(M)$  is countable. We can also assume that  $K$  is algebraically closed and uncountable; if not, then tensor  $D_n$  and  $M$  over  $K$  with an extension of  $K$ . Let  $\phi : D_n(K, f) \rightarrow \text{End}_K(M)$  be the map induced by the module action.

We induct on  $n$ . The case  $n = 0$  is trivial. Assume then that  $n > 0$ . Suppose that  $d(M) < n$ . By lemma 4.4, there exists  $\alpha \in K$  such that  $\psi := \phi(x_n - \alpha)$  is non-invertible. By lemma 4.5, we can set  $t = \sigma(x_n) = x_n - \alpha$ . Hence  $\psi$  is either not injective or injective but not surjective. Let  $M'$  and  $M''$  be the kernel and cokernel of  $\psi$  respectively. Set  $A = D_{n-1}$  and observe that  $\psi$  is an  $A$ -homomorphism.

Suppose that  $\psi$  is injective but not surjective. Since  $M$  is finitely generated, it has a standard filtration  $M_i$ . This induces a standard filtration  $M''_i$  on the cokernel  $M''$ . Observe that  $M_r/tM_{r-1}$  maps surjectively into  $M''_r$  and  $\psi$  is injective. Hence, as  $r \rightarrow \infty$  we get

$$\dim_K(M''_r) \leq \dim_K(M_r) - \dim_K(M_{r-1}) = O(r^{d(M)-1}).$$

Recall that  $M'' \neq 0$ , so it contains some  $\eta \neq 0$  which is in  $M''_{r_0}$  for some  $r_0 \geq 0$ . We now define the  $A$ -module  $L'' = A\eta$  and a filtration  $L''_i = A_i\eta$ . By induction, we have  $n - 1 \leq d(L'')$ . It is clear that  $L''_r \subset (D_n)_r\eta \subset M''_{r+r_0}$ , therefore

$$\dim_K(L''_r) \leq \dim_K(M''_{r+r_0}) = O(r^{d(M)-1})$$

as  $r \rightarrow \infty$  and  $d(L'') \leq d(M) - 1$ . By assumption,  $d(M) < n$ , so we get the contradiction  $n - 1 \leq d(L'') < n - 1$ .

Suppose that  $\psi$  is not injective. Let  $N$  be the union of  $\ker(\phi(t^m))$  for  $m > 0$ . This is an  $A$ -submodule of  $M$  which is stable under  $\phi(t)$ . We will show that it is also stable under  $\phi(\partial/\partial t)$ ; recall that

$$(\partial/\partial t)t^m = t^m(\partial/\partial t) + mt^{m-1}$$

Let  $\gamma \in N$ ; then  $t^m\gamma = 0$  for some  $m > 0$ . We have

$$t^{m+1}(\partial/\partial t)\gamma = (\partial/\partial t)t^m\gamma - mt^{m-1}\gamma = 0$$

and so  $N$  is stable under  $\phi(\partial/\partial t)$ . By lemma 4.3,  $N$  is finitely generated and thus  $d(N) < n$ . Also,  $N \neq 0$  since  $M' \subset N$ . We can therefore replace  $M$  by  $N$  and assume without loss of generality that for all  $\gamma \in M$ , there exists  $m > 0$  such that  $t^m\gamma = 0$ .

We want that  $\ker(\phi(\partial/\partial t - \alpha)) = 0$  for all  $\alpha \in K$ . Suppose not, and let  $\gamma \neq 0$  be an element of this kernel for some  $\alpha$  such that  $m$  is minimal. The relation  $(\partial/\partial t)\gamma = \alpha\gamma$  gives us

$$0 = (\partial/\partial t)t^m\gamma = \alpha t^m\gamma + mt^{m-1}\gamma = mt^{m-1}\gamma.$$

Since  $\text{char}(K) = 0$ , we have that  $t^{m-1}\gamma = 0$ , contradicting the minimality of  $m$ .

Finally, let  $\tau$  be the map from lemma 4.5 with  $\alpha, \beta = 0$ ; let  $M_\tau$  be  $M$  with this different module action. Using the above arguments for  $M_\tau$  instead of  $M$ , we have that  $M_\tau$  is non-zero and finitely generated over  $D_n$  with  $d(M_\tau) < n$  and  $\ker(\phi(t - \alpha)) = 0$ . Recall that  $\phi(t - \alpha)$  is not a unit of  $\text{End}_K(M_\tau)$  by lemma 4.4. Using the map  $\sigma$  from lemma 4.5 to get  $M_{\sigma\tau}$  by replacing  $t - \alpha$  with  $t$ , we recover the earlier case where  $M' = 0$ ; this was shown to be a contradiction.  $\square$

*Proof of 4.1 part 2.* Assume that  $d = n$  and consider the ascending chain of  $D_n$ -submodules

$$0 = L_0 \subset L_1 \subset \cdots \subset L_k.$$

For each  $i$ , choose  $\gamma_i \in L_i$  such that  $\gamma_i$  has non-zero image in  $L_i/L_{i-1}$ . Let  $M_i$  be the  $D_n$ -module generated by  $\gamma_1, \dots, \gamma_i$ . We now have the ascending chain

$$0 = M_0 \subset M_1 \subset \cdots \subset M_k$$

where  $M_i$  and  $M_i/M_{i-1}$  are finitely generated for all  $i$ . By the first half of the theorem and remark 3.1, we have that  $n \leq d(M_i/M_{i-1}) \leq d(M_i)$  for all  $i$ . Since  $M_i$  is a  $D_n$ -submodule of  $M$  with an  $(n, e)$ -filtration, we have  $d(M_i) \leq n$ . Thus,  $n = d(M_i/M_{i-1}) = d(M_i)$  for all  $i$ . This implies, by remark 3.1, that

$$e(M_i) = e(M_{i-1}) + e(M_i/M_{i-1}) > e(M_{i-1})$$

for all  $i$ . Observe that  $e(M_1) > 0$ , so  $e(M_i) > i$  for all  $i$ . Since  $M_k$  is a submodule of  $M$  and  $d(M_k) = n$ , we have  $e(M_k) \leq e$ . Hence,  $k \leq e$ .  $\square$

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