The Euler Totient, the Möbius and the Divisor Functions

Rosica Dineva

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Mount Holyoke College
South Hadley, MA 01075

Acknowledgements

1   Introduction

The theory of numbers is an area of mathematics which deals with the properties of whole and rational numbers. Analytic number theory is one of its branches, which involves study of arithmetical functions, their properties and the interrelationships that exist among these functions. In this paper I will introduce some of the three very important examples of arithmetical functions, as well as a concept of the possible operations we can use with them. There are four propositions which are mentioned in this paper and I have used the definitions of these arithmetical functions and some Lemmas which reflect their properties, in order to prove them.

2   Definitions

Here are some definitions to illustrate how the functions work and describe some of their most useful properties.

2.1   Arithmetical function

A real or complex valued function with domain the positive integers is called an *arithmetical* or a number-theoretic function.

2.2   Multiplicative functions

An arithmetical function f is called *multiplicative* if f is not identically zero and if $f(mn) = f(m)f(n)$ whenever $(m, n) = 1$. A multiplicative function f is called *completely multiplicative* if $f(mn) = f(m)f(n)$ for all m, n.

2.3 The Möbius function

The *Möbius function* is an arithmetical function, which takes the following values:

$$\mu(1) = 1$$

and for $n = p_1^{a_1} * p_2^{a_2} * ... * p_m^{a_m}$, where $n > 1$, we define $\mu(n)$ to be:

$\mu(n) = (-1)^m \qquad$ if $a_1 = a_2 = ... = a_m = 1$,

$\mu(n) = 0 \qquad\qquad$ otherwise.

This definition implies that the Möbius function will be zero if and only if n has a square factor larger than one. Let us look at a short table of the values of $\mu(n)$ for some positive integers:

n	$\mu(n)$
1	1
2	-1
3	-1
4	0
5	-1
6	1
7	-1
8	0
9	0
10	1

The Möbius function is an example of a multiplicative, but not completely multiplicative function, since $\phi(4) = 0$ but $\phi(2)\phi(2) = 1$. One if its most

important applications is in the formulas for the *Euler totient*, which is the next function I will define.

2.4 The Euler totient

The *Euler totient function* is defined to be the number of positive integers which are less or equal to an integer and are relatively prime to that integer:

for $n \geq 1$, the Euler totient $\phi(n)$ is:

$$\phi(n) = \sum_{k=1}^{n} {}'1,$$

where the ' indicates that the sum is only over the integers relatively prime to n. Below is a table of the values of $\phi(n)$ for some small positive integers:

n	$\phi(n)$
1	1
2	1
3	2
4	2
5	4
6	2
7	6
8	4
9	6
10	4

There is a formula for the divisor sum which is one of the most useful properties of the Euler totient:

Lemma 1: for $n \geq 1$ we have

$$\sum_{d|n} \phi(d) = n.$$

Since the Euler totient is the number of positive integers relatively prime to n we can calculate $\phi(n)$ as a product over the prime divisors of n, where $n \geq 1$:

Lemma 2:

$$\phi(n) = n * \prod_{p|n} (1 - \frac{1}{p}).$$

The following formula gives a relation between the Euler totient and the Möbius function:

Lemma 3: for $n \geq 1$ we have:

$$\phi(n) = \sum_{d|n} \mu(d) \frac{n}{d}.$$

The Euler totient is another multiplicative function which is not completely multiplicative because $\phi(4) = 2$ but $\phi(2)\phi(2) = 1$.

2.5   The divisor functions

For a real or a complex number α and an integer $n \geq 1$ we define

$$\sigma_\alpha(n) = \sum_{d|n} d^\alpha$$

to be the sum of the αth powers of the divisors of n, called the divisor function $\sigma_\alpha(n)$. These functions are also multiplicative.

If we look at the trivial case when $\alpha = 0$ we say that $\sigma_0(n)$ is the number of divisors of n. In the case that $\alpha = 1$ we define $\sigma_1(n)$ as the sum of the divisors of n. Since the function is multiplicative we know that for $n = p_1^{a_1} p_2^{a_2} ... p_m^{a_m}$ then $\sigma_\alpha(n) = \sigma_\alpha(p_1^{a_1} \sigma_\alpha(p_2^{a_2}) ... \sigma_\alpha(p_m^{a_m})$. There is a formula for the divisor function of an integer power of a prime:

Lemma 3:

$$\sigma_\alpha(p^a) = 1^\alpha + p^\alpha + p^{2\alpha} + ... + p^{a\alpha} = \frac{p^{\alpha(a+1)} - 1}{p^\alpha - 1} \; if \; \alpha \neq 0$$

$$\sigma_0(p^a) = a + 1 \; if \; \alpha = 0$$

The next definition I will introduce is the *Dirichlet product of arithmetical functions*, which is represented by a sum, occurring very often in number theory.

2.6   Dirichlet product of arithmetical functions

The *Dirichlet product* of two arithmetical functions f and g is defined to be an arithmetical function $h(n)$ such that:

$$(f * g)(n) = h(n) = \sum_{d|n} f(d) g(\frac{n}{d}).$$

If we look at the formula for the relation between the Euler totient and the Möbius function, we will see that for a function N, such that $N(n) = n$ then $\phi = \mu * N$.

3 Propositions and their proofs

3.1 Proposition 1

For a positive integer n we have that:

$$\frac{n}{\phi(n)} = \sum_{d|n} \frac{\mu^2(d)}{\phi(d)}$$

where the sum is over all the divisors of n.

Proof: We know by Lemma 2 that

$$\phi(n) = n * \prod_{p|n} (1 - \frac{1}{p})$$

so if we let $n = p_1^{a_1} * p_2^{a_2} * .... * p_m^{a_m}$, we can express $\phi(n)$ in the following

$$\phi(n) = n * (1 - \frac{1}{p_1}) * (1 - \frac{1}{p_2}) * .... * (1 - \frac{1}{p_m})$$

Taking a common denominator for each of the terms in the parentheses we see that:

$$\phi(n) = \frac{n * (p_1 - 1) * (p_2 - 1) * .... * (p_m - 1)}{p_1 * p_2 * .... * p_m}.$$

Thus we have that

$$\frac{n}{\phi(n)} = \frac{n}{\frac{n*(p_1-1)*(p_2-1)*.......*(p_m-1)}{p_1*p_2*.......*p_m}} = \frac{p_1 * p_2 * .... * p_m}{(p_1 - 1) * (p_2 - 1) * .... * (p_m - 1)}$$

This equation is our result for the left hand side of the identity we have to prove. We will denote $\frac{n}{\phi(n)}$ with the initials LHS for the rest of the proof.

Now we look at the right hand side of the identity above. From the definition of the Möbius function we know that for $n = p_1^{a_1} * p_2^{a_2} * \ldots\ldots * p_m^{a_m}$ $\mu(n) = (-1)^m$ and $\mu(n) = 0$, when n has a square term. Therefore $\mu^2(d) = 1$ if d has no square term and $\mu^2(d) = 0$ if d has a square term. Thus our sum will be over only the square free divisors of n since if a divisor is not square free we will have a zero term. For the rest of this paper d_1 will represent a divisor of n which is square free:

$$\sum_{d_1 | n} \frac{\mu^2(d_1)}{\phi(d_1)} = \sum_{d_1 | n} \frac{1}{\phi(d_1)}$$

Since d_1 is square free d_1 will be any product of the prime factors of n, where each prime could be used only once in the prime factorization of each divisor d_1. The last statement means that d_1 takes on each of the values $1, p_1, \ldots, p_m, p_1 * p_2, p_1 * p_3, \ldots, p_{m-1} * p_m, p_1 * p_2 * p_3, \ldots\ldots\ldots, p_1 * p_2 * \ldots * p_m$. The RHS will thus become

$$\sum_{d_1 | n} \frac{1}{\phi(d_1)} = \frac{1}{\phi(1)} + \frac{1}{\phi(p_1)} + \ldots\ldots + \frac{1}{\phi(p_1 * p_2)} + \ldots.. + \frac{1}{\phi(p_1 * p_2 * \ldots.. * p_m)}$$

$$\sum_{d_1 | n} \frac{1}{\phi(d_1)} = 1 + \frac{1}{p_1 - 1} + \ldots. + \frac{1}{(p_1 - 1) * (p_2 - 1)} + \ldots.. + \frac{1}{(p_1 - 1) * \ldots.. * (p_m - 1)}$$

The common denominator of this sum will be $(p_1-1)*(p_2-1)*\ldots..(p_m-1)$, so after we get the sum over a common denominator the right hand side becomes:

$$RHS = \frac{(p_1 - 1) * \ldots * (p_m - 1) + (p_2 - 1) * \ldots * (p_m - 1) + \ldots. + (p_m - 1) + \ldots. + 1}{(p_1 - 1) * (p_2 - 1) * \ldots * (p_m - 1)}$$

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We can now rearrange the terms in the numerator, starting with the last one and for the numerator of the right hand side, RHS_N, we get the following:

$$RHS_N = 1+(p_1-1)+...+(p_m-1)+(p_1-1)*(p_2-1)+....+(p_1-1)*(p_2-1)*...*(p_m-1)$$

When we look carefully at each of the terms in this equation we can see that each term is actually the ϕ function of some prime or of some product of primes. We can therefore rewrite the numerator as:

$$RHS_N = \phi(1) + \phi(p_1) + ... + \phi(p_m) + \phi(p_1*p_2) + ...+ \phi(p_1*p_2*...*p_m),$$

Thus for the RHS_N we find that:

$$RHS_N = \sum_{l|p_1*p_2*...*p_m} \phi(l)$$

By Lemma 1 we know that:

$$\sum_{d|n} \phi(d) = n,$$

thus when $n = p_1p_2...p_m$ we have the $RHS_N = p_1p_2...p_m$. The right hand side of the identity we want to prove becomes $RHS = \frac{p_1*...*p_m}{(p_1-1)*...*(p_m-1)}$ and since this equals the LHS, our identity is proven:

$$\frac{n}{\phi(n)} = \sum_{d|n} \frac{\mu^2(d)}{\phi(d)}.$$

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3.2 Proposition 2

For all n with at most 8 distinct prime factors we have that $\phi(n) > \frac{n}{6}$.

Proof: We will first prove the proposition for an n with 8 distinct prime factors. Let $n = p_1^{a_1} p_2^{a_2} ... p_8^{a_8}$ so using Lemma 2 for the Euler totient we see that:

$$\phi(n) = p_1^{a_1} p_2^{a_2} ... p_8^{a_8} (1 - \frac{1}{p_1})(1 - \frac{1}{p_2})...(1 - \frac{1}{p_8})$$

Since $p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8$ are distinct prime factors we know that $p_1 \geq 2, p_2 \geq 3, p_3 \geq 5, p_4 \geq 7, p_5 \geq 11, p_6 \geq 13, p_7 \geq 17, p_8 \geq 19$, because these are the first 8 distinct primes. Therefore $\frac{1}{p_1} \leq \frac{1}{2}, ..., \frac{1}{p_7} \leq \frac{1}{17}, \frac{1}{p_8} \leq \frac{1}{19}$ and then $(1 - \frac{1}{p_1}) \geq (1 - \frac{1}{2}), ...(1 - \frac{1}{p_8}) \geq (1 - \frac{1}{19})$. We can now substitute in the equation for $\phi(n)$ and since we will substitute each term in parentheses with a term which is less or equal to the initial one we will get:

$$\phi(n) \geq n(1 - \frac{1}{2})...(1 - \frac{1}{19}),$$

therefore

$$\phi(n) \geq n\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \frac{10}{11} \cdot \frac{12}{13} \cdot \frac{16}{17} \cdot \frac{18}{19},$$

and then $\phi(n) \geq n.\frac{1658880}{9699690}$, so $\phi(n) \geq 0,171n$. But $\frac{n}{6} \approx 0,167n$, which means that $\phi(n) > \frac{n}{6}$ for $n = p_1^{a_1} ... p_8^{a_8}$.

Each of the factors $(1 - \frac{1}{p_1}), (1 - \frac{1}{p_2}), ..., (1 - \frac{1}{p_8})$ is less that one since the smallest possible prime is two, so each of the terms in parentheses will be less than one, which means that when we multiply the product by it, we will decrease its value. So if our integer n has less than 8 distinct prime factors

the value for its Euler totient will be greater than the value of the Euler totient of an integer with 8 distinct prime factors. Thus, we have proved that for all integers n with 8 or less distinct prime factors

$$\phi(n) > \frac{n}{6}.$$

3.3   Proposition 3

Let $f(x)$ be defined for all rational x in $0 \leq x \leq 1$ and let

$$F(n) = \sum_{k=1}^{n} f(\frac{k}{n})$$

$$F^*(n) = \sum_{\substack{k=1 \\ (k,n)=1}}^{n} f(\frac{k}{n})$$

Then

A)$F^* = \mu * F$, the Dirichlet product of μ and F.

Proof: Let us look at the Dirichlet product of the two functions.

$$\mu * F = \sum_{d|n} \mu(d) F(\frac{n}{d}) = \sum_{d|n} \mu(d) \sum_{k=1}^{\frac{n}{d}} f(\frac{kd}{n}).$$

Again, since we have the Möbius function all divisors d, which are not square free will give us zero for the sum. We will denote the square free divisors by d_1 and let $n = p_1^{a_1} p_2^{a_2} ... p_m^{a_m}$. Then the divisors d_1 will be all the primes $p_1, ..., p_m$ and all the possible products of these primes. The Möbius function will take the values -1 and 1, when we have odd and even number of primes in our divisors, respectively. Then our Dirichlet product will become:

$$\mu * F = \sum_{d_1 | n} \mu(d_1) \sum_{k=1}^{\frac{n}{d_1}} f\left(\frac{kd_1}{n}\right) = A$$

$$A = F(n) - F\left(\frac{n}{p_1}\right) - \ldots - F\left(\frac{n}{p_m}\right) + F\left(\frac{n}{p_1 p_2}\right) + \ldots + F\left(\frac{n}{p_{m-1} p_m}\right) - \ldots + (-1)^m F(1),$$

and when we substitute with the formula we have for $F(n)$ we get:

$$A = \sum_{k=1}^{n} f\left(\frac{k}{n}\right) - \sum_{k=1}^{\frac{n}{p_1}} f\left(\frac{kp_1}{n}\right) - \ldots - \sum_{k=1}^{\frac{n}{p_m}} f\left(\frac{kp_m}{n}\right) + \sum_{k=1}^{\frac{n}{p_1 p_2}} f\left(\frac{kp_1 p_2}{n}\right) + \ldots$$

$$+ \sum_{k=1}^{\frac{n}{p_{m-1} p_m}} f\left(\frac{kp_{m-1} p_m}{n}\right) - \ldots + (-1)^m \sum_{k=1}^{\frac{n}{p_1 \cdots p_m}} f\left(\frac{kp_1 \ldots p_m}{n}\right).$$

We know that

$$F^*(n) = \sum_{\substack{k=1 \\ (k,n)=1}}^{n} f\left(\frac{k}{n}\right) = \sum_{k=1}^{n} f\left(\frac{k}{n}\right) - \sum_{\substack{k=1 \\ (k,n) \neq 1}}^{n} f\left(\frac{k}{n}\right)$$

Let us look at the second term in the last difference. We have that $(k, n) \neq 1$ which means that k will have at least one of the prime factors of n. Thus, $k \in \{p_i b_i | i \in [1, m], b_i \in Z^*\}$ and since we know that $k \leq n$, then $p_i b_i \leq n$ and therefore for each value of i, $b_i \leq \frac{n}{p_i}$. We can then rewrite the second term in our difference as the sum of the sums with each of the b_is as a variable. But since we are letting b_i be anything less than $\frac{n}{p_i}$ each of the sums will count the factors with more than one distinct prime factor of n in the numerator of $f(\frac{k}{n})$ twice. This observation will be true for all the subsequent sums, too - the sums which start with two distinct prime factors of n in the variable b_i will count all the other sums with three prime distinct factors of

n twice and so on. This way when we are writing out the sum and start with the first group of sums - the ones for which b_i has at least one of the distinct prime factors of n - we will have to subtract the second group of sums - with 2 or more of the distinct prime factors of n, which on its own will subtract all the sums with 3 or more of the distinct prime factors of n, thus we will have to add those sums and then we would have added twice the next sums and so on. Therefore for the second term we will get an alternating series of sums, so that we can account for all the integers less than n, which have common factors with n and get rid of all the terms which repeat. Thus,

$$\sum_{\substack{k=1 \\ (k,n)\neq 1}}^{n} f(\frac{k}{n}) = \sum_{k=1}^{\frac{n}{p_1}} f(\frac{kp_1}{n}) + ... + \sum_{k=1}^{\frac{n}{p_m}} f(\frac{kp_m}{n}) - \sum_{k=1}^{\frac{n}{p_1 p_2}} f(\frac{kp_1 p_2}{n}) - ...$$

$$- \sum_{k=1}^{\frac{n}{p_{m-1}p_m}} f(\frac{kp_{m-1}p_m}{n}) + ... + (-1)^m \sum_{k=1}^{\frac{n}{p_1...p_m}} f(\frac{kp_1...p_m}{n})$$

We can now substitute in the formula we derived for $F^*(n)$ and it will become:

$$F^*(n) = \sum_{\substack{k=1 \\ (k,n)=1}}^{n} f(\frac{k}{n}) = \sum_{k=1}^{n} f(\frac{k}{n}) - [\sum_{k=1}^{\frac{n}{p_1}} f(\frac{kp_1}{n}) + ... + \sum_{k=1}^{\frac{n}{p_m}} f(\frac{kp_m}{n}) - \sum_{k=1}^{\frac{n}{p_1 p_2}} f(\frac{kp_1 p_2}{n}) - ...$$

$$- \sum_{k=1}^{\frac{n}{p_{m-1}p_m}} f(\frac{kp_{m-1}p_m}{n}) + ... + (-1)^m \sum_{k=1}^{\frac{n}{p_1...p_m}} f(\frac{kp_1...p_m}{n})]$$

Opening the parenthesis and applying the negative sign in front of them we will get:

$$F^*(n) = \sum_{k=1}^{n} f(\frac{k}{n}) - \sum_{k=1}^{\overline{\frac{n}{p_1}}} f(\frac{kp_1}{n}) - ... - \sum_{k=1}^{\overline{\frac{n}{p_m}}} f(\frac{kp_m}{n}) + \sum_{k=1}^{\overline{\frac{n}{p_1 p_2}}} f(\frac{kp_1 p_2}{n}) + ...$$

$$+ \sum_{k=1}^{\overline{\frac{n}{p_{m-1} p_m}}} f(\frac{kp_{m-1}p_m}{n}) - ... + (-1)^m \sum_{k=1}^{\overline{\frac{n}{p_1...p_m}}} f(\frac{kp_1...p_m}{n}),$$

which is exactly our result for $\sum_{d_1|n} \mu(d_1) \sum_{k=1}^{\overline{\frac{n}{d_1}}} f(\frac{kd_1}{n}) = \mu * F$. We have proved the proposition.

B)$\mu(n)$ is the sum of the primitive nth roots of unity:

$$\mu(n) = \sum_{\substack{k=1 \\ (k,n)=1}}^{n} e^{2\pi i k/n}$$

Proof: Let $f(x) = e^{2\pi i x}$ - this is a valid function for $f(x)$ because the exponential function is defined for all rational x satisfying the condition on x. Then for $n = p_1^{a_1} p_2^{a_2}...p_m^{a_m}$:

$$F(n) = \sum_{k=1}^{n} e^{2\pi i \frac{k}{n}}$$

$$F^*(n) = \sum_{\substack{k=1 \\ (k,n)=1}}^{n} e^{2\pi i \frac{k}{n}}$$

Let us look at the right hand side of the identity we want to prove. We will denote the square free divisors of n d_1 and the divisors with squares in them d_2, where $d_1 \neq n$ and $d_2 \neq n$. We can use part A of Proposition 3 and substitute in the formula for the Dirichlet product. Thus, since $F^*(n) = (\mu * F)(n)$:

$$F^*(n) = \sum_{d|n} \mu(d)F(\frac{n}{d}) = \mu(n)F(1) + \sum_{d_2|n} \mu(d_2)F(\frac{n}{d_2}) + \sum_{d_1|n} \mu(d_1)F(\frac{n}{d_1})$$

When the argument of the Möbius function has a square its value is 0. Therefore, the sum over the square divisors is zero. Let us look at the other terms in F^*. For $n = 1$

$$F(1) = \sum_{k=1}^{n} e^{2\pi i k} = e^{2\pi i} = cos(2\pi) + isin(2\pi) = 1 + 0 = 1,$$

thus $F^*(n) = \mu(n) + \sum_{d_1|n} \mu(d_1)F(\frac{n}{d_1})$.

We will now solve the second term in our equation.

$$F(\frac{n}{d_1}) = \sum_{k=1}^{\frac{n}{d_1}} e^{2\pi i \frac{d_1 k}{n}} = e^{2\pi i \frac{d_1}{n}} + e^{2\pi i \frac{2d_1}{n}} + ... + e^{2\pi i \frac{d_1(\frac{n}{d_1}-2)}{n}} + e^{2\pi i \frac{d_1(\frac{n}{d_1}-1)}{n}} + e^{2\pi i \frac{n}{n}}$$

We want to prove that $\mu(n) = F^*(n)$, where $f(x) = e^{2\pi i x}$, which means that we need to prove that $F(\frac{n}{d_1}) = 0$. In order to do this we will multiply $F(\frac{n}{d_1})$ by a function which is not of value 1 but will still return the same function. In our case $e^{2\pi i \frac{d_1}{n}} \neq 1$, since $d_1 \neq n$ by the definition of d_1. Thus,

$$e^{2\pi i \frac{d_1}{n}} F(\frac{n}{d_1}) = e^{2\pi i \frac{d_1}{n}} \sum_{k=1}^{\frac{n}{d_1}} e^{2\pi i \frac{d_1 k}{n}} = \sum_{k=1}^{\frac{n}{d_1}} e^{2\pi i \frac{(k+1)d_1}{n}} = A,$$

and summing over we get

$$A = e^{2\pi i \frac{2d_1}{n}} + e^{2\pi i \frac{3d_1}{n}} + ... + e^{2\pi i \frac{d_1(\frac{n}{d_1}-2+1)}{n}} + e^{2\pi i \frac{d_1(\frac{n}{d_1}-1+1)}{n}} + e^{2\pi i \frac{d_1(\frac{n}{d_1}+1)}{n}}$$

If we look carefully at the terms of both products we will see that if we rearrange the terms and let the last term of the second product become the first the two products will be completely the same. Then, $e^{2\pi i \frac{d_1}{n}} F(\frac{n}{d_1}) = F(\frac{n}{d_1})$, which means that $e^{2\pi i \frac{d_1}{n}} F(\frac{n}{d_1}) - F(\frac{n}{d_1}) = 0$ and we know that $e^{2\pi i \frac{d_1}{n}} \neq 1$ by the definition of d_1 and since the only solutions to $F(\frac{n}{d_1})(e^{2\pi i \frac{d_1}{n}} - 1) = 0$ are $e^{2\pi i \frac{d_1}{n}} = 1$ or $F(\frac{n}{d_1}) = 0$, therefore $F(\frac{n}{d_1}) = 0$.

So then the right hand side of the identity we want to prove becomes $F^*(n) = \mu(n) + \sum_{d_2|n} 0.F(\frac{n}{d_2}) + \sum_{d_1|n} \mu(d_1).0 = \mu(n)$ and by the identity we have from part A of Proposition 3 we have thus proven

$$\mu(n) = \sum_{\substack{k=1 \\ (k,n)=1}}^{n} e^{2\pi i k/n}.$$

3.4 Proposition 4

For $n \geq 1$ we have

$$\sigma_1(n) = \sum_{d|n} \phi(d)\sigma_0(\frac{n}{d}).$$

Proof: Let $n = p_1^{a_1} * p_2^{a_2} * ... * p_m^{a_m}$. By the definition of the divisor function σ_α and the Euler totient we know that they are both multiplicative which means that, for example, $\sigma_1(n) = \sigma_1(p_1^{a_1} * p_2^{a_2} * ... * p_m^{a_m}) = \sigma_1(p_1^{a_1}) * \sigma_1(p_2^{a_2}) * ... * \sigma_1(p_m^{a_m})$. From the properties of the divisor function it follows that $\sigma_\alpha(p^a) = \frac{p^{\alpha(a+1)}-1}{p^\alpha - 1}$ when $\alpha \neq 0$ and $\sigma_\alpha(p^a) = a+1$ when $\alpha = 0$. Therefore in our case when $\alpha = 1$ we solve the equation:

$$\sigma_1(n) = \sigma_1(p_1^{a_1} * p_2^{a_2} * ... * p_m^{a_m}) = \sigma_1(p_1^{a_1}) * \sigma_1(p_2^{a_2}) * .... * \sigma_1(p_m^{a_m}) =$$

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$$\frac{p_1^{a_1+1} - 1}{p_1 - 1} * \frac{p_2^{a_2+1} - 1}{p_2 - 1} * \ldots\ldots * \frac{p_m^{a_m+1} - 1}{p_m - 1}$$

We will call this side the left hand side, LHS. Let us look at the right hand side, RHS, of the equation. We see that our variable in this case is d, which represents the divisors of n. These divisors will have the form $d = p_1^{i_1} * p_2^{i_2} * \ldots * p_m^{i_m}$ where $0 \leq i_1, i_2, \ldots\ldots, i_r \geq a_1, a_2, \ldots\ldots, a_r$, respectively. This form of each of the divisors will allow us to represent them all since this way we will be able to count for the divisors with different primes and the different powers these primes could have. Then when we substitute for d the RHS will become

$$RHS = \sum_{d|n} \phi(d)\sigma_0(\frac{n}{d}) = \sum_{i_1=0}^{a_1} \sum_{i_2=0}^{a_2} \ldots\ldots \sum_{i_m=0}^{a_m} \phi(p_1^{i_1} * p_2^{i_2} * \ldots * p_m^{i_m})\sigma_o(\frac{n}{p_1^{i_1} * p_2^{i_2} * \ldots * p_m^{i_m}})$$

for which be the definition of n and since the Euler totient is a multiplicative function we get the following result:

$$RHS = \sum_{i_1=0}^{a_1} \ldots\ldots \sum_{i_m=0}^{a_m} \phi(p_1^{i_1}) * \ldots * \phi(p_2^{i_2})\sigma_0(\frac{p_1^{a_1} * p_2^{a_2} * \ldots * p_m^{a_m}}{p_1^{i_1} * p_2^{i_2} * \ldots * p_m^{i_m}})$$

and when we divide in the argument of the divisor function we generate

$$RHS = \sum_{i_1=0}^{a_1} \ldots\ldots \sum_{i_m=0}^{a_m} \phi(p_1^{i_1}) * \ldots * \phi(p_2^{i_2})\sigma_0(p_1^{a_1-i_1} * p_2^{a_2-i_2} * \ldots * p_m^{a_m-i_m})$$

and again because the divisor function is also multiplicative we can write the equation and rearrange its terms so that

$$RHS = \sum_{i_1=0}^{a_1} \ldots\ldots \sum_{i_m=0}^{a_m} \phi(p_1^{i_1})\sigma_0(p_1^{a_1-i_1}) * \phi(p_1^{i_2})\sigma_0(p_2^{a_2-i_2}) * \ldots * \phi(p_m^{i_m})\sigma_0(p_m^{a_m-i_m})$$

We have m-sums and for each one of them only two terms of the equations are variable - the ones whose variables are the respective i_j, where $j = 1, ..., r$. We can then rearrange the terms so that each couple of terms will be summed over in the appropriate sum. We are allowed to do that since for each sum only terms change and the rest $m - 1$ couples of terms are constants, which we can get in front of the sum. Applying this rule m times we rearrange the RHS:

$$RHS = \sum_{i_1=0}^{a_1} \phi(p_1^{i_1})\sigma_0(p_1^{a_1-i_1}) \sum_{i_2=0}^{a_2} \phi(p_2^{i_2})\sigma_0(p_2^{a_2-i_2}) ... \sum_{i_m=0}^{a_m} \phi(p_m^{i_m})\sigma_0(p_m^{a_m-i_m})$$

The functions for each sum are the same, the only difference being that they depend on a different prime number. The variables for these functions assume the same values, so it will be enough to solve for one of these sums and then this result will apply to all the other sums taking into account their respective primes. Let us take the first sum and solve for it:

We will denote the first sum with A, therefore

$$\sum_{i_1=0}^{a_1} \phi(p_1^{i_1})\sigma_0(p_1^{a_1-i_1}) = A$$

and then

$$A = \phi(1)\sigma_0(p_1^{a_1}) + \phi(p_1)\sigma_0(p_1^{a_1-1}) + ... + \phi(p_1^{a_1-2})\sigma_0(p_1^2) + \phi(p_1^{a_1-1})\sigma_0(p_1) + \phi(p_1^{a_1})\sigma_0(1)$$

when we apply the formulas we know for the functions in question we get

$$A = a_1 + 1 + (p_1-1)(a_1-1+1) + p_1(p_1-1)(a_1-2+1) + p_1^2(p_1-1)(a_1-3+1) +$$

$$+p_1^3(p_1-1)(a_1-4+1)+...+p_1^{a_1-1-1}(p_1-1)(a_1-(a_1-1)+1)+p_1^{a_1-1}(p_1-1)$$

we can then factor out their common factor so

$$A = a_1 + 1 + (p_1 - 1)(a_1 + p_1(a_1 - 1) + p_1^2(a_1 - 2) + ... + 2p_1^{a_1-2} + p_1^{a_1-1})$$

and when we open the brackets inside

$$A = a_1 + 1 + (p_1 - 1)(a_1 + p_1 a_1 - p_1 + p_1^2 a_1 - 2p_1^2 + p_1^3 a_1 - 3p_1^3 + ... + 2p_1^{a_1-2} + p_1^{a_1-1})$$

we now multiply the two terms in the brackets and so

$$A = a_1 + 1 + (a_1 p_1 - a_1 + a_1 p_1^2 - a_1 p_1 - p_1^2 + p_1 + a_1 p_1^3 - a_1 p_1^2 - 2p_1^3 + 2p_1^2 + a_1 p_1^4 -$$

$$-a_1 p_1^3 - 3p_1^4 + 3p_1^3 + ... + 3p_1^{a_1-2} - 3p_1^{a_1-3} + 2p_1^{a_1-1} - 2p_1^{a_1-2} + p_1^{a_1} - p_1^{a-1})$$

we can see that some terms repeat but with opposite signs, so these terms will give 0. Some other terms can be combined together so when we apply all operations possible we will end up with

$$A = a_1 + 1 + (-a_1 + p_1 + p_1^2 + p_1^3 + p_1^4 + ... + p_1^{a_1-3} + p_1^{a_1-2} + p_1^{a_1-1} + p_1^{a_1})$$

so when we add the first term the sum comes out to be

$$A = 1 + p_1 + p_1^2 + p_1^3 + p_1^4 + ... + p_1^{a_1-3} + p_1^{a_1-2} + p_1^{a_1-1} + p_1^{a_1}$$

and we can see that this is a geometric progression so when we use the formula for a geometric progression the first sum equals

$$A = \sum_{i_1=0}^{a_1} \phi(p_1^{i_1})\sigma_0(p_1^{a_1-i_1}) = \frac{1-p_1^{a_1+1}}{1-p_1} = \frac{p_1^{a_1+1}-1}{p_1-1}$$

Since this result will apply to all m sums we solve the right hand side to be:

$$RHS = \sum_{i_1=0}^{a_1} \phi(p_1^{i_1})\sigma_0(p_1^{a_1-i_1}) \sum_{i_2=0}^{a_2} \phi(p_2^{i_2})\sigma_0(p_2^{a_2-i_2})... \sum_{i_m=0}^{a_m} \phi(p_m^{i_m})\sigma_0(p_m^{a_m-i_m})$$

so when we substitute the result we got the product becomes

$$RHS = \frac{p_1^{a_1+1} - 1}{p_1 - 1} * \frac{p_2^{a_2+1} - 1}{p_2 - 1} * ... * \frac{p_{r-1}^{a_{r-1}+1} - 1}{p_{r-1} - 1} * \frac{p_r^{a_r+1} - 1}{p_r - 1}$$

which means that

$$LHS = RHS$$

so we proved the equality

$$\sigma_1(n) = \sum_{d|n} \phi(d)\sigma_0(\frac{n}{d}).$$

References

[1] Tom M. Apostol, *Introduction to Analytic Number Theory*, Springer-Verlag, New York, Inc, USA, 1976