

# IGUSA LOCAL ZETA FUNCTION

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## 1. THE P-ADIC NUMBERS AND P-ADIC VALUATION

One of the most commonly used number systems are the field of integers  $Z$  and the field of rational numbers  $Q$ . We construct the field of the real numbers  $R$ , which is contained in the rational numbers by adding all the limits of the Cauchy sequences. In a very similar way we can construct the field of p-adic rational numbers  $Q_p$  and p-adic integers  $Z_p$ , where we will use a different absolute value defined in the theorem below, so the sequences in the rational numbers and the integers will be different but will still be Cauchy sequence so we will be able to take their limits to construct the two new fields. We will first give a definition of the order of a number in p-adic valuation.

*Definition 1.1:* The order of a number  $x$ ,  $ord_p(x)$ , where  $x = p^\alpha \frac{a}{b}$ ,  $p$  is a prime and  $a$  does not divide  $b$ , is the highest power of  $p$  which divides  $x$ . So in other words in our case  $ord_p(x) = \alpha$ . In the case that  $p$  does not divide  $x$ , then  $ord_p(x) = 1$  and the last case is  $ord_p(0) = \infty$ . We can now state a theorem, which will give us a definition of the properties of the absolute values in different number systems and the we will be able to define the form of a p-adic rational number of a p-adic integer:

**Ostrowski Theorem:**

$$\begin{aligned}
 (1) \quad |x|_\infty &= \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases} \\
 (2) \quad |x|_0 &= \begin{cases} 0, & x=0 \\ 1, & x \neq 0 \end{cases} \\
 (3) \quad |x|_p &= \begin{cases} 0, & x=0 \\ p^{-ord_p(x)}, & x \neq 0 \end{cases}
 \end{aligned}$$

We can observe from the definition of the theorem that the p-adic absolute value of a number will be small if the number is heavily divisible by the prime and relatively big if there are a lot of primes in the denominator of the number. We can now define what a p-adic number looks like:

*Definition 1.2:* Every number  $x$  can be uniquely represented by:

$x = b_0p^{-m} + b_1p^{-m-1} + \dots + b_{m-1}p + b_m + b_{m+1}p + b_{m+2}p^2 + \dots$ , where  $b_i \in \{0, 1, 2, \dots, p-1\}$ , so then if  $b_0 \neq 0$   $|x|_p = p^m$ . For a p-adic integer we won't have rational parts of its representation and the sequence will be a finite one, so then if  $y$  is a p-adic integer -  $y = a_0 + a_1p + a_2p^2 + \dots + a_mp^m$ , where again  $a_0 \in \{0, 1, 2, \dots, p-1\}$  and if  $a_0 \neq 0$  then  $y$  is called a unit and its p-adic valuation is 1.

Here are a few examples:

$$\text{ex.1: } |12|_2 = |2^2 \cdot 3|_2 = 2^{-2} = \frac{1}{4}$$

$$\text{ex.2: } \left| \frac{27}{125} \right|_5 = \left| \frac{27}{5^3} \right|_5 = 5^3 = 125$$

$$\text{ex.3: } |32|_3 = |2^5|_3 = 1$$

## 2. THE HAAR MEASURE

The field of p-adic integers  $Z_p$  can be decomposed into cosets modulo the maximal ideal  $pZ_p$  in  $Z_p$  and then we can represent  $Z_p$  as the disjoint union of all of these cosets:

$$Z_p = 0 + pZ_p \cup 1 + pZ_p \cup \dots \cup (p-1) + pZ_p = \prod_{a=0}^{p-1} a + pZ_p$$

All cosets with the exception of  $0 + pZ_p$  have absolute value 1 so we can also represent the field of p-adic integers as:

$Z_p = pZ_p \cup (Z_p - pZ_p) = pZ_p \cup U_p$ , where  $U_p$  is the field of the units in  $Z_p$ . We then call an interval  $I$  the set of the form  $I = \alpha + p^e Z_p$ , where

$\alpha \in Z_p$  and  $e \geq 0$ . The Haar measure is the way we measure these intervals and it is the maximal distance between any two points in the interval. We denote this measure by  $m(I)$  and we write  $\int_I dx = m(I)$ . The Haar measure has four properties which we will use when doing integration over the p-adic integers. These are:

- (1)  $m(I) \geq 0$  and  $m(\emptyset) = 0$
- (2) if  $I_1$  and  $I_2$  are two different intervals and  $I_1 \cap I_2 = \emptyset$ , then  $m(I_1 \cup I_2) = m(I_1) + m(I_2)$
- (3) let  $\beta$  be a fixed p-adic integer, then  $m(\beta + I) = m(I)$  - the measure is translation invariant
- (4)  $m(Z_p) = 1$ .

### 3. IGUSA LOCAL ZETA FUNCTION

We define the Igusa local zeta function as:

$$z(s) = \int_{Z_p^n} |f(x)|_p^s dx \quad \text{for } \operatorname{Re}(s) > 0,$$

where  $f(x) = f(x_1, x_2, \dots, x_n)$  is a polynomial of  $n$  variables with coefficients in the p-adic integers and  $s$  is a complex variable. For the rest of this paper we will denote  $p^{-s} = t$  for clarification purposes.

In 1975, Jun-ichi Igusa has proved that the local zeta function is a fractional function of the variable  $t$ . Using the Haar measure we can solve these integrals, because unlike the ordinary defined integrals we need to multiply the answer by the measure of the set over which we are integrating. The method for solving the Igusa local zeta function I am going to introduce in this paper is the Stationary Phase Formula, which is discussed in the next section.

## 4. STATIONARY PHASE FORMULA (SPF)

We can compute the local zeta functions by SPF, using the following formula:

$$z(s) = (p^n - |N|)p^{-n} + (|N| - |S|)p^{-n}t\left(\frac{1 - p^{-1}}{1 - p^{-1}t}\right) + \int_{x \in S} |f(x)|_p^s dx,$$

where

$f(x)$  is an  $n$ -variable polynomial

$$\bar{f}(x) \equiv f(x) \pmod{p}$$

$|N|$  is the number of vectors  $x$  in  $F_p^n$  for which  $\bar{f}(x) \equiv 0 \pmod{p}$

$|S|$  is the number of singular vectors  $x$  in  $N$  such that all partial derivatives of  $\bar{f}$  at  $x$  are congruent to 0 mod  $p$

We can rely on the stationary phase formula for all local zeta functions except the so-called degenerate cases. We define  $f(x)$  of a local zeta function to be degenerate if there are non singular vectors in  $N$ , such that all partial derivatives of  $f(x)$  at  $x$  are congruent to 0 mod  $p$ . In this case, SPF won't give us the right answer and we have to use other methods or exclude the primes for which the polynomial is degenerate.

## 5. EXAMPLE OF A LOCAL ZETA FUNCTION IN TWO DIMENSIONS

The example I am going to solve is a function of two variables:  $f(x) = x^3 + x^2y^2 + y^6$ . When we solve for the partial derivatives, we get  $3x^2 + 2xy^2 \equiv 0 \pmod{p}$  and  $6y^5 + 2x^2y \equiv 0 \pmod{p}$  simultaneously. Then the possible values when the derivatives are 0 are  $(0,0)$  or when  $31y^4 \equiv 0$  for all  $y$  and this happens when  $p = 31$  because then for all different values of  $x, y$  the partial derivatives are 0 mod 31. Therefore, in this case our polynomial will be degenerate so SPF will not work. In

order to solve it i will assume that  $p \neq 31$ . Let us look at the different components of our formula - the number of singular points when the primes is not 31 is only (0,0). So then  $|S| = 1$ . The values for  $|N|$  are more difficult to find. During the research I did on Igusa local zeta function I used a program called "Polygusa", which was created by Kathleen Hoornaert for her thesis. This program allowed me to enter a function and a specific prime, and it calculated the Igusa local zeta function as a function of  $s$ . I entered my function in the program and calculated the values for all primes under 130 except 31. From the results for the different primes I computed all  $|N|$  values and looked for a certain pattern so that I can solve the Igusa local zeta function for an arbitrary prime. The results I generated are in the table below - the first column represents all prime numbers under 100 for which  $|N| = 1$ , the second one contains all primes with the value of  $|N| = p$  and the third one all the ones for which there is some specific formula for the cardinality of the set  $N$ . There were only two primes under 130, for which the  $|N|$  was not 1 or  $p$  - for these primes there was a distinct formula for the set  $N$ . In both cases  $|N| = 3p - 2$ . Here is the table:

$ N  = 1$	$ N  = p$	<i>Other</i>
2	3	47
5	11	67
7	13	
19	17	
41	23	
59	29	
71	37	
97	43	
101	53	
103	61	
107	73	
109	79	
113	83	
	89	
	127	

## 6. CONCLUSION

This summer I explored Igusa local zeta functions, the methods to solve them and the cases when these methods do not always work. I concentrated on an example for which the SPF method fails and tried to find some formula and correlation for the primes for which the function becomes degenerate. This example is possibly part of a big class of examples for which the SPF method does not work and we need to find some other way of computing the function.