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LOCAL ZETA FUNCTIONS OF DEGENERATE POLYNOMIALS AND POLES ASSOCIATED WITH DEGENERACY

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Abstract. We examine cases in which a polynomial f is degenerate with respect to its Newton polyhedron and when a pole results from degeneracy. We focus on polynomials which are reducible into linear factors, in particular those which are degenerate for all primes p with respect to the improper face of their Newton polyhedra. Two examples for which a new pole arises from degeneracy are computed and motivation for further research of degenerate polynomials is given.

1. Introduction

In this paper, we focus on the Igusa local zeta function for a particular class of degenerate polynomials, in particular those that factor into linear homogeneous polynomials with specific conditions placed upon the singular points modulo p, where p a prime. We compute Igusa's local zeta function for two examples in which a new pole occurs due to the degeneracy of the polynomial and discuss how its Newton polyhedron can be used to determine when the pole arises.

2. Preliminaries

We begin by introducing basic properties of p-adic numbers and their construction, along with topological and algebraic properties which expose their utility. We focus specifically on valuations and absolute values on the rational numbers $\mathbb{Q}$. Further exposition of valuations and absolute values on general fields may be found in [3] and [9].

Definition 2.1. If K is a field, $x, y \in K$, we define a *valuation* v to be a function such that:

(1) $v(xy) = v(x) + v(y)$

(2) $v(x + y) \geq min\{v(x), v(y)\}$

Definition 2.2. Let p be a prime. We define the *p-adic valuation* to be the function $v_p : \mathbb{Q} \longrightarrow \mathbb{Z}\backslash\{0\}$ such that for $x \in \mathbb{Q}$,

$$x = p^{v_p(x)}x',$$

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$x' = \frac{a}{b}$, $a, b \in \mathbb{Z}$ where $p \nmid ab$.

Definition 2.3. For $x \in \mathbb{Q}$, the *p-adic absolute value* is defined as follows:

$$|x|_p = \begin{cases} p^{-v_p(x)} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases} \tag{1}$$

Definition 2.4. A sequence $\{x_i\}_{i=1}^{\infty}$ is a *Cauchy sequence* if for every $\epsilon > 0$ there exists an integer N such that if $m, n \geq N$, $|x_n - x_m| < \epsilon$.

In order to construct the p-adic numbers, we want to complete $\mathbb{Q}$ with respect to the newly defined absolute value, so here we make reference to the analogy of the p-adic numbers with the real numbers $\mathbb{R}$. Note that there exist sequences of rational numbers which do not converge in $\mathbb{Q}$, so in order to obtain a field in which all Cauchy sequences converge, we essentially add all limits of Cauchy sequences to the elements of $\mathbb{Q}$ in order to obtain the real numbers. We now do the same with respect to the p-adic absolute value and define the p-adic numbers formally as equivalence classes of Cauchy seqences of rational numbers.

Definition 2.5. Let S denote the set of all Cauchy sequences of rational numbers and let S_0 denote the set of Cauchy sequences of rational numbers converging to 0. Then the p-adic numbers $\mathbb{Q}_p$ are formally defined as follows:

$$\mathbb{Q}_p = S/S_0$$

Since we have defined the p-adic absolute value on $\mathbb{Q}$, we want to extend it to all of $\mathbb{Q}_p$ so we do this by considering a representative of the equivalence class of a given element in the aforementioned quotient ring. Thus we define the absolute value of any p-adic number x in the following way:

$$|x|_p = \lim_{n\to\infty} |x_n|_p$$

where $\{x_n\}$ is a representative of x in $\mathbb{Q}_p$.

Definition 2.6. An element $x \in \mathbb{Q}_p$ is called a *p-adic integer* if $|x|_p \leq 1$. We denote by $\mathbb{Z}_p^*$ the set

$$\mathbb{Z}_p^* = \{x \in \mathbb{Q}_p \mid |x| = 1\}$$

and refer to these elements as *p-adic units.*

Remark 2.7. There is an equivalent characterization of the p-adic numbers which allows each element to be represented as a finite-tailed 'Laurent expansion' of the following form:

$$x = a_{-m}p^{-m} + a_{-(m-1)}p^{-(m-1)} + \cdots + a_0 + a_1 p + a_2 p^2 + \cdots$$

where $0 \leq a_i \leq p-1$, and $a_{-m} \neq 0$. Thus we may view the absolute value of an element $x \in \mathbb{Q}_p$ as being $|x|_p = p^{-k}$ where k is the highest power of p that can be factored out of such an expansion. In other words, $|x|_p$ is equal to p^{-i} where a_i is the coefficient of the leftmost nonzero term in the p-adic expansion.[3]

Now viewing the p-adic integers in this context, we see that all p-adic integers must have $a_i = 0$ for all $i < 0$ and all p-adic units must have the same property with the additional characteristic that $a_0 \neq 0$. So if $x \in \mathbb{Z}_p$, an expansion of x is given by:

$$x = a_0 + a_1 p + a_2 p^2 + \cdots + a_m p^m + \cdots$$

with $0 \leq a_i \leq p-1$ for all i.

3. p-adic Analysis and the Igusa Local Zeta Function

We now focus on integration over $\mathbb{Z}_p^n$, but before we can define such an integral, we must introduce the measure associated with $\mathbb{Q}_p$.

Proposition 3.1. *$\mathbb{Q}_p$ is a locally compact topological group.*

A proof of this fact relies on the topology of $\mathbb{Q}_p$, in particular that $\mathbb{Z}_p$ is a compact neighborhood about 0.[3] For now we use Proposition 1.8 to assert the existence of a Haar measure m on $\mathbb{Z}_p$ which is unique up to a positive real constant. We list important properties of this Haar measure below defining $m(E)$ to be $\int_{\mathbb{Z}_p} \chi_E dm$ for $E \subset \mathbb{Z}_p$ and χ_E the characteristic function.

(1) $m(\emptyset) = 0$

(2) If $a \in \mathbb{Z}_p$, then $m(a + E) = m(E)$.

(3) If $A \cap B = \emptyset$, then $m(A \cup B) = m(A) + m(B)$.

(4) $m(\mathbb{Z}_p) = 1$

Definition 3.2. The *Igusa local zeta function* associated to f is defined to be the integral:

$$Z(s) = \int_{\mathbb{Z}_p^n} |f(x_1, \ldots, x_n)|^s dx_1 \cdots dx_n,$$

where $s \in \mathbb{C}, Re(s) > 0$, and by convention $t = p^{-s}$.[8]

In general there are various methods used to compute local zeta functions including resolution of singularities which was used by Jun-ichi Igusa to prove rationality of the local zeta function in 1975. [6] He used the fact that a resolution of singularities exists for any polynomial over a field of characteristic zero which was proven by Hironaka in 1964.[4] Here we focus on two methods, namely the Stationary Phase Formula introduced by Igusa in 1994, and a formula for Igusa's local zeta function introduced by Kathleen Hoornaert.[1], [7]

Theorem 3.3. *(Stationary Phase Formula): Let* $N_1 = \{x \in \mathbb{F}_p^n \mid f(x) \equiv 0 \pmod{p}\}$ *and* $S = \{\alpha \in N_1 \mid \frac{\partial f}{\partial x_i}(\alpha) \equiv 0 \pmod{p}$*, for all* $i\}$. *Then:*

$$Z(s) = p^{-n}(p^n - |N_1|) + \frac{(|N_1| - |S|)p^{-n}t(1-p^{-1})}{1-p^{-1}t} + \sum_{\alpha \in S} \int_{\alpha + p\mathbb{Z}_p^n} |f(x)|^s dx$$

To illustrate how the Stationary Phase Formula is used, we provide an example for which the formula is applied once.

Example 3.4. Let $f(x, y) = xy - y^2$. We must determine $|N_1|$, so setting f congruent to 0 modulo p, we have:

$$xy - y^2 \equiv 0 \pmod{p}.$$

Thus if $y \neq 0$, we multiply by y^{-1} to obtain the simpler equation $x \equiv y \pmod{p}$, so we have p solutions to this equation. If $y = 0$, then we may choose any x since the variable y is in both terms of f. Thus adding the cardinalities of these solutions together, excluding the intersection point $(0, 0)$, we have $|N_1| = 2p - 1$. Now examining the partial derivatives given below

$$\frac{\partial f}{\partial x} \equiv y \pmod{p}$$

$$\frac{\partial f}{\partial y} \equiv x - 2y \pmod{p}$$

we see that since y must be 0 modulo p, the second equation implies that x must also be 0 so the only singular point is $(0, 0)$. Thus SPF gives the following:

$$Z(s) = p^{-2}(p^2 - (2p-1)) + \frac{((2p-1)-1)p^{-2}t(1-p^{-1})}{1-p^{-1}t} + \int_{p\mathbb{Z}_p^2} |xy - y^2|^s dxdy$$

$$= p^{-2}(p^2 - 2p + 1) + \frac{(2p-2)p^{-2}t(1-p^{-1})}{1-p^{-1}t} + p^{-2} \int_{\mathbb{Z}_p^2} |(px_1)(py_1) - p^2 y_1^2|^s dx_1 dy_1$$

$$= p^{-2}(p^2 - 2p + 1) + \frac{(2p-2)p^{-2}t(1-p^{-1})}{1-p^{-1}t} + p^{-2}t^2 \int_{\mathbb{Z}_p^2} |x_1 y_1 - y_1^2|^s dx_1 dy_1$$

since the change in measure for the singular integral is $dxdy = p^{-2}dx_1dy_1$ and we factor out $|p^2| = p^{-2s} = t^2$. Notice that the last integral in the third line is the original zeta function, so we have the following equation:

$$Z(s) = p^{-2}(p^2 - 2p + 1) + \frac{(2p-2)p^{-2}t(1-p^{-1})}{1-p^{-1}t} + p^{-2}t^2 Z(s),$$

so grouping terms under a common denominator and moving all multiples of $Z(s)$ to the left, we have:

$$\begin{aligned} Z(s) &= \frac{(1-p^{-1})^2(1+p^{-1}t)}{(1-p^{-1}t)(1-p^{-2}t^2)} \\ &= \frac{(1-p^{-1})^2}{(1-p^{-1}t)^2} \end{aligned}$$

We now introduce the preliminaries necessary to understand Kathleen Hoornaert's formula given in her Ph.D. thesis.[1]

Definition 3.5. Let $f = \sum_{k=(k_1,\ldots,k_n)\in\mathbb{N}^n} a_k x_1^{k_1}\cdots x_n^{k_n} \in \mathbb{Z}[x_1,...,x_n]$ with $f(0) = 0$ and let $\mathbb{R}^+$ denote the set of all nonnegative real numbers. Define the support of f by $supp(f) = \{k \in \mathbb{N}^n \mid a_k \neq 0\}$, and let $\Gamma'(f)$ denote the convex hull of $supp(f)$. Then the Newton polyhedron denoted $\Gamma(f)$ is given by the following equation $\Gamma(f) = \Gamma'(f) + (\mathbb{R}^+)^n$.

Definition 3.6. A *proper face* of the Newton polyhedron is the intersection of a supporting hyperplane H and the Newton polyhedron of f such that $H \cap int(\Gamma(f)) = \emptyset$ where *int* denotes the interior. The *improper face,* denoted $\Gamma(f)$, is the entire Newton polyhedron. We call a face τ a *facet* if the codimension of τ is 1.

Definition 3.7. Let $f = \sum_{k=(k_1,\ldots,k_n)} a_k x_1^{k_1}\cdots x_n^{k_n} \in \mathbb{Z}[x_1,...,x_n]$ with $f(0) = 0$. Then we define:

$$m(a) = \inf_{x\in\Gamma(f)}\{a\cdot x\}$$

and

$$F(a) = \{x \in \Gamma(f) \mid a\cdot x = m(a)\},$$

which we refer to as the *first meet locus of a.*

Definition 3.8. Let τ be a face of $\Gamma(f)$. Then the *cone associated to* τ is:

$$\Delta_\tau = \{a \in (\mathbb{R}^+)^n \mid F(a) = \tau\}.$$

For our purposes, we may view the cone Δ_τ as the strictly positive span of the normals to all facets containing τ. For example if $\tau = \{(x,0) \in \mathbb{R}^2 \mid x \geq 1\}$, then the normal to τ is $(0,1)$ so $\Delta_\tau = \{\lambda(0,1) \mid \lambda > 1\}$.

4. Degenerate Polynomials in $\mathbb{Z}[x_1, ..., x_n]$ and their Local Zeta Functions

Definition 4.1. For a polynomial $f \in \mathbb{Z}[x_1, ..., x_n]$, we define

$$f_\tau = \sum_{k \in supp(f) \cap \tau} a_k x_1^{k_1} \cdots x_n^{k_n},$$

where $k = (k_1, ..., k_n)$ and a_k is determined by f.

Definition 4.2. A polynomial $f(x_1, ..., x_n) \in \mathbb{Z}[x_1, ..., x_n]$ is *non-degenerate* with respect to its Newton polyhedron if for every face τ, the system of congruences:

$$f_\tau \equiv 0 \pmod p$$

$$\frac{\partial f_\tau}{\partial x_i} \equiv 0 \pmod p$$

has no solution in $(\mathbb{F}_p^*)^n$.

We now introduce a formula for computing the Igusa local zeta function for non-degenerate polynomials and then compute an example for which we can apply this formula.

Theorem 4.3. *Let f be as in Definition 3.5 with $f(0) = 0$. Suppose f is non-degenerate with respect to its Newton polyhedron $\Gamma(f)$. Then define $N_\tau, L_\tau,$ and S_{Δ_τ} by the following:*

$$N_\tau = \{x \in (\mathbb{F}_p^*)^n \mid f_\tau(x) \equiv 0 \pmod p)\}$$

$$L_\tau = p^{-n}((p-1)^n - \frac{p|N_\tau|(p^s - 1)}{p^{s+1} - 1})$$

$$S_{\Delta_\tau} = \sum_{k \in \mathbb{N}^n \cap \Delta_\tau} p^{-\sigma(k) - m(k)s}$$

where $\sigma(k) = \sum_{i=1}^n k_i$ if $k = (k_1, ..., k_n)$.
Then

$$Z(s) = \sum_{\tau \in \Gamma(f)} L_\tau S_{\Delta_\tau}.$$

Furthermore, if the cone Δ_τ *is strictly positively spanned by some linearly independent vectors* $a_1, ..., a_r \in \mathbb{N}^n$*, then*

$$S_{\Delta_\tau} = \frac{\sum_h p^{\sigma(h)+m(h)s}}{(p^{\sigma(a_1)+m(a_1)s} - 1)\cdots(p^{\sigma(a_r)+m(a_r)s} - 1)}$$

where h *is indexed from the following set:*

$$\mathbb{N}^n \bigcap \left\{ \sum_{i=1}^{r} \lambda_i a_i \mid 0 \leq \lambda_i < 1 \text{ for } 1 \leq i \leq r \right\}.$$

Example 4.4. We compute the local zeta function for the non-degenerate polynomial $f(x,y) = x + y^3$ using the formula given in Theorem 4.3. To verify that f is indeed non-degenerate, we consider the faces of $\Gamma(f)$:

$$\begin{gathered}
\tau_1 = \{(0,y) \in \mathbb{R}^2 \mid y \geq 3\} \\
\tau_2 = \{(0,3)\} \\
\tau_3 = \{(1-t)(0,3) + t(1,0) \mid 0 \leq t \leq 1\} \\
\tau_4 = \{(1,0)\} \\
\tau_5 = \{(x,0) \mid x \geq 1\} \\
\tau_6 = \Gamma(f)
\end{gathered}$$

We can use Definition 4.1 to find each f_τ given below:

$$\begin{gathered}
f_{\tau_1} = f_{\tau_2} = y^3 \\
f_{\tau_3} = f_{\tau_6} = x + y^3 \\
f_{\tau_4} = f_{\tau_5} = x
\end{gathered}$$

so notice that the partial derivatives at x for x and $x+y^3$ are always 1, implying there are no singular points in $(\mathbb{F}_p^*)^2$. It is clear that if y^3 is set equal to 0 modulo p, then there are no nonzero solutions. Thus f is non-degenerate so we apply Theorem 4.3 to obtain the following sum:

$$Z(s) = \sum_{\tau \in \Gamma(f)} L_\tau S_{\Delta_\tau}$$

$$= (p^{-2}(p-1)^2)(\frac{1}{p-1}) + (p^{-2}(p-1)^2)(\frac{1}{(p-1)(p^{4+3s}-1)})$$

$$+(p^{-2}((p-1)^2 - \frac{p(p-1)(p^s-1)}{p^{s+1}-1}))(\frac{1}{p^{4+3s}-1})$$

$$+(p^{-2}(p-1)^2)(\frac{1+p^{2+s}+p^{3+2s}}{(p^{4+3s}-1)(p-1)}) + (p^{-2}(p-1)^2)(\frac{1}{p-1})$$

$$+(p^{-2}((p-1)^2 - \frac{p(p-1)(p^s-1)}{p^{s+1}-1}))(1)$$
$$= \frac{1-p^{-1}}{1-p^{-1}t}$$

Since the formula in Theorem 4.3 assumes f is non-degenerate, we must determine how to use a similar computation to find the local zeta function of degenerate polynomials. In particular, we follow the proof of this formula in Hoornaert's thesis until the condition of non-degeneracy is needed. We begin by partitioning $\mathbb{Z}_p^n$ into annuli and then use the fact that the cones Δ_τ of $\Gamma(f)$ partition $(\mathbb{R}^+)^n$, in particular the cones partition the lattice points in $(\mathbb{R}^+)^n$.

$$Z(s) = \int_{\mathbb{Z}_p^n} |f(x)|^s dx$$
$$= \sum_{k\in\mathbb{N}^n} \int_{p^{k_1}\mathbb{Z}_p^*\times\cdots\times p^{k_n}\mathbb{Z}_p^*} |f(x)|^s dx$$
$$= \sum_{\tau} \sum_{k\in\mathbb{N}^n\cap\Delta_\tau} \int_{p^{k_1}\mathbb{Z}_p^*\times\cdots\times p^{k_n}\mathbb{Z}_p^*} |f(x)|^s dx$$
$$= \sum_{\tau} \sum_{k\in\mathbb{N}^n\cap\Delta_\tau} p^{-\sigma(k)} \int_{(\mathbb{Z}_p^*)^n} |f(u_1,...,u_n)|^s du_1\cdots du_n$$
$$= \sum_{\tau} \sum_{k\in\mathbb{N}^n\cap\Delta_\tau} p^{-\sigma(k)-m(k)s} \int_{(\mathbb{Z}_p^*)^n} |\tilde{f}(u_1,...,u_n)|^s du_1\cdots du_n$$

The term $p^{-\sigma(k)}$ comes from the change in measure $dx_1\cdots dx_n = p^{-\sigma(k)}du_1\cdots du_n$, and the term $p^{-m(k)s}$ comes from factoring out the highest common power of p in the integrand $f(u_1,...,u_n)$, and then taking its p-adic absolute value raised to the s power. The new integrand $\tilde{f}$ is the result of factoring out this power of p, and for non-degenerate polynomials, this integrand is not dependent on the cone of τ since there are no singular points in $(\mathbb{F}_p^*)^n$, that is, when SPF is applied to the integral, we have the closed formula given by L_τ. So it is possible to multiply this integral with the sum $\sum_{k\in\mathbb{N}^n\cap\Delta_\tau} p^{-\sigma(k)-m(k)s}$. However, if the polynomial f is degenerate, this integral will be dependent on $k \in \mathbb{N}^n \cap \Delta_\tau$. Using this last equation, we can compute the local zeta function by splitting the sum in Theorem 4.3 as follows:

Equation 4.5.

$$Z(s) = \sum_{\tau non-deg.} L_\tau S_{\Delta_\tau} + \sum_{\tau deg.} \sum_{k\in\mathbb{N}^n\cap\Delta_\tau} p^{-\sigma(k)-m(k)s} \int_{(\mathbb{Z}_p^*)^n} |\tilde{f}(u_1,...,u_n)|^s du_1\cdots du_n$$

This equation is identical to Kathleen Hoornaert's sum in the proof of Theorem 3.3 before the condition of non-degeneracy is used. Using this equation, we will now focus on the case when there are singular points modulo p in $(\mathbb{F}_p^*)^n$. In particular, we restrict attention to the class of degenerate polynomials which factor into linear homogeneous polynomials $a_1(x_1, ..., x_n), ..., a_m(x_1, ..., x_n)$ with certain conditions placed upon the singular points.

Condition 1: Let $a_i(x_1, ..., x_n) = \sum_{j=1}^{n} c_{ij}x_j$ with $c_{ij} \in \mathbb{Z}$ such that $p \nmid c_{ij}$ for any i, j, and $f = \prod_{i=1}^{m} a_i(x_1, ..., x_n), m \geq 2$. Then suppose there exists a simultaneous zero $b \in (\mathbb{F}_p^*)^n$ to two linear factors a_{c_1}, a_{c_2}, some c_1, c_2 with $1 \leq c_1, c_2, \leq m$.

Condition 2: Let f be as in Condition 1, and let S denote the set of singular points modulo p. Suppose every point $s \in S$ is a zero modulo p of every linear factor of f.

Condition 3: The spanning vector of the cone of every facet of $\Gamma(f)$ in Condition 2 is of the form $(\epsilon_1, ..., \epsilon_n)$, where $\epsilon_i = 0$ or $1, 1 \leq i \leq n$.

Proposition 4.6. *If Condition 1 is satisfied, then f is degenerate with respect to the improper face of its Newton polyhedron.*

Proof. If Condition 1 is satisfied, then f must vanish at b since it is the product of all linear factors. Furthermore, we have the following for the partial derivatives of f:

$$\left\{ \frac{\partial f}{\partial x_j} \equiv \frac{\partial a_1}{\partial x_j} a_2 \cdots a_m + a_1 \frac{\partial a_2}{\partial x_j} a_3 \cdots a_m + \cdots + a_1 \cdots a_{m-1} \frac{\partial a_m}{\partial x_j} \equiv 0 \pmod{p} \right\}$$

for $1 \leq j \leq n$, so b also solves this system since either a_{c_1} or a_{c_2} appear in each summand. Thus the partial derivatives vanish at b and the condition for degeneracy is satisfied since the whole polynomial corresponds to the improper face.

□

Lemma 4.7. *In Condition 2 is satisfied, then $|S| = p^{n-1}$ or p^{n-2}.*

Proof. Let $M_i = \{x \in \mathbb{F}_p^n \mid a_i(x) \equiv 0 \pmod{p}\}$. Then $N_1 = \bigcup_{i=1}^{m} M_i$. Furthermore, if we choose M_i, M_j, for some i, j, we claim that their intersection is S, the set of singular points modulo p of f. To verify this, let $s \in S$. By Condition 2, s is in any such intersection since it satisfies all linear factors modulo p. Now let s in $M_i \cap M_j$. By the proof of Proposition 3.6., s is a singular point so this intersection must be the entire set S. Now we consider two cases:

Case 1: There exist i, j, k such that the coefficient of x_k in a_i and a_j is nonzero where $1 \leq i, j \leq m, 1 \leq k \leq n$.

Let $M = M_i \cap M_j$ for some i, j. M is the set of all singular points, so it suffices to count M in order to determine $|S|$. If $M_i = M_j$, then $|S| = p^{n-1}$ since we may choose a variable with coefficient nonzero, say x_l, so that there are p^{n-1} choices for the other $n-1$ variables and we solve for x_l so it is fixed. Otherwise, we may solve for x_k since p does not divide all coefficients. This gives the following equation:

$$x_k = \frac{-c_{i1}x_1 - c_{i2}x_2 - \cdots - c_{ik}\hat{x_k} - \cdots c_{in}x_n}{c_{ik}},$$

where we delete the k^{th} term in the numerator.
Thus setting the component of x_k in each set M_i, M_j equal, we obtain the intersection M, so since we then have a linear equation with the same $n-1$ variables, we may solve it to obtain all points in M so there are p^{n-2} choices for all other variables.

Case 2: For all $k, 1 \leq k \leq n$, there do not exist i, j such that the coefficient of x_k in a_i and a_j is nonzero, $1 \leq i, j \leq m$.

Take M_i, M_j for some i, j. Then if a_i is a linear term in r variables and a_j is a linear term in s variables, solve for the r^{th} and s^{th} variable to obtain the intersection of M_i and M_j given below:

$$M_i \cap M_j = \{(x_1, ..., g_1(x_1, x_2..., x_{r-1}), y_1, ..., g_2(y_1, y_2, ..., y_{s-1})) \mid x_i, y_i \in \mathbb{F}_p\}$$

Thus choosing any value for $x_1, ..., x_{r-1}, y_1, ..., y_{s-1}$, we see there are p^{n-2} elements in this set.

□

In order to examine what poles result from degeneracy, we consider the case when Condition 2 is satisfied and then address the case when Condition 2 is not satisfied. In Section 4, an example will be provided which is subsumed in the case when Condition 2 is satisfied, so we consider this case in the following proposition.

Proposition 4.8. *Let f satisfy Condition 2 and Condition 3. Then a pole occurs in the local zeta function for f which results from the sum corresponding to the degenerate faces of the Newton polyhedron in Equation 3.5.*

Proof. We apply SPF to determine the local zeta function for f. Since all singular points are zeros of all linear factors of f modulo p, SPF gives the following for A, B dependent on $|N_1|$:

$$Z(s) = A + B + \sum_{\alpha\in S}\int_{\alpha+p\mathbb{Z}_p^n}|f(x)|^s dx$$
$$= A + B + \sum_{\alpha\in S} p^{-n}\int_{\mathbb{Z}_p^n}|f(\alpha+px)|^s dx$$
$$= A + B + \sum_{\alpha\in S} p^{-n}t^m Z(s)$$
$$= A + B + |S|p^{-n}t^m Z(s)$$

since each singular point α satisfies all linear factors and each factor is homogeneous. Thus we have a pole depending on $|S|$ and the numerator of $Z(s)$. Ultimately, we also want to show that this pole does not result from the sum over the non-degenerate faces of $\Gamma(f)$, so we verify that:

(1) $|S|$ is a power of p using Lemma 4.7

(2) The denominator $1 - |S|p^{-n}t^m$ yields a pole, that is, the numerator of $Z(s)$ does not cancel with all factors of this denominator

(3) The pole given by $1 - |S|p^{-n}t^m$ does not come a candidate pole.

(1) By Lemma 4.7, $|S| = p^{n-1}$ or p^{n-2}, so the pole we obtain is of the form $1 - p^{-1}t^m$ or $1 - p^{-2}t^m$. Thus there is a potential pole at $s_0 = \frac{-1}{m}$ or $\frac{-2}{m}$. To determine if s_0 is an actual pole, we must verify that it does not occur in the numerator of $Z(s)$.

(2) When we get A and B over a common denominator, the numerator will have at most a linear term in t, so we must check that $1 - p^{-1}t^m$ and $1 - p^{-2}t^m$ do not factor in $\mathbb{Q}[t]$ in such a way that the pole is cancelled by the numerator, that is, a linear term in t divides either $1 - p^{-1}t^m$ or $1 - p^{-2}t^m$. However, if either of these terms is divisible by a linear term, then since $\mathbb{Q}$ is an integral domain, this implies that either $1 - p^{-1}t^m$ or $1 - p^{-2}t^m$ would have a zero in $\mathbb{Q}$, but this cannot happen since for the first term it implies:

$$1 - p^{-1}t^m = 0$$

or equivalently t is an m^{th} root of p. So t is only a rational number when $m = 1$, a contradiction to assumption.

The second term $1 - p^{-2}t^m$ has a root if and only if:

$$1 - p^{-2}t^m = 0$$

that is, t is an m^{th} root of p^2, which occurs only when $m = 2$. However if $m = 2$, there are two factors of f, and since the exponent of p is -2, these factors are distinct, so we compute the terms A and B using SPF. Since we add two solution sets and exclude the intersection to compute $|N_1|$, we have $|N_1| = 2p^{n-1} - p^{n-2}$. Furthermore, since by the proof of Lemma 3.7, the set of singular points modulo p, denoted S, is exactly the intersection mentioned previously, $|S| = p^{n-2}$.

Hence we have the following for the local zeta function:

$$Z(s) = p^{-n}(p^n - 2p^{n-1} + p^{n-2}) + \frac{(2p^{n-1} - 2p^{n-2})p^{-n}t(1 - p^{-1})}{(1 - p^{-1}t)(1 - p^{-2}t^2)}$$

which implies that

$$Z(s) = \frac{(1 - p^{-1})^2(1 - p^{-1}t)}{1 - p^{-1}t} + \frac{2p^{-1}t(1 - p^{-1})^2}{(1 - p^{-1}t)(1 - p^{-2}t^2)}.$$

So after combining common terms in the numerator we have:

$$Z(s) = \frac{(1 - p^{-1})^2(1 + p^{-1}t)}{(1 - p^{-1}t)(1 - p^{-1}t)(1 + p^{-1}t)}$$

so the term $1 + p^{-1}t$ cancels and we are left with the same pole. Thus in all cases, a new pole appears in the denominator.

(3) Now we want to show that the pole obtained in SPF does not occur as a candidate pole, that is, it does not result from the denominators of any S_{Δ_τ} for non-degenerate τ. So we consider the facets of $\Gamma(f)$ since by Theorem 4.3, the denominators of each S_{Δ_τ} are products of the form

$$(p^{\sigma(a_1)+m(a_1)s} - 1) \cdots (p^{\sigma(a_v)+m(a_v)s} - 1)$$

where each a_i is the normal to the i^{th} facet of $\Gamma(f)$. In addition, we consider the case when a candidate pole factors into a linear term which arises from SPF, which happens only in the case where the new pole s_0 comes from the term $1 - p^{-2}t^2$, that is when $m = 2$.

Assuming Condition 3, we want to show that the value $m(a_i)$ is strictly less than m, unless $\sigma(a_i) = n$. So using an equivalent characterization of Definition 2.7,

$$m(a_i) = \min_{x \in supp(f)} \{a_i \cdot x\}.$$

Hence since our polynomial f is homogeneous of degree m, each element of the support has components whose sum is m, so suppose $a_i \neq (1, 1, ..., 1)$. Then there exists j such that the j^{th} component of a_i is zero. Now we claim that there exists an element of the support y such that the j^{th} component of y is nonzero. However this is true since all variables $x_1, ..., x_n$ occur in the product of f so in particular since x_j is in the product, there will be a point of the support with j^{th} component nonzero. Thus

$$a_i \cdot (\epsilon_1, ..., \epsilon_n) < m$$

since this dot product is the sum of all the components of y except for a nonzero component. Thus we have that $m(a_i) < m$, so none of these denominators are equal to $1 - p^{-1}t^m$ or $1 - p^{-2}t^m$. If $a_i = (1, 1, ..., 1)$, then $m(a_i)$ is exactly m since for all points in the support, the sum of all components is m since f is homogeneous. Thus since $\sigma(1, 1, ..., 1) = n$, this a_i would result in the denominator $1 - p^{-n}t^m$, but for s_0, the contributing denominator has a power of p strictly less than n, so this does not result in the pole s_0.

We must also consider the case when the denominator of an S_{Δ_τ} has a factor of $1 - p^{-1}t$ since this might result in the new pole. However, this is the case when $m = 2$, so the candidate pole must come from a denominator of the form $1 - p^{-l}t^2$ by above arguments since $m(a_i) < m$. Thus, since $m = 2$, this only occurs for $l = n = 2$, so there are only two variables. If the zeros of each linear factor are the same, then the pole is $1 - p^{-1}t^2$ which does not factor and does not come from a candidate pole by arguments above. If the zeros of each linear factor are different, then there are no zeros in $(\mathbb{F}_p^*)^2$ so f does not satisfy Condition 1.

Thus the pole s_0 is contributed by the sums over degenerate τ.

□

5. A Detailed Example for which a New Pole Arises from Degenerate Faces

Example 5.1. As an example of the type of degenerate polynomial introduced in Proposition 4.8, consider the polynomial $f(x, y, z) = (x - y)(x + z)(y + z)$. By Proposition 4.6, since two of the linear factors in $f, x - y$ and $x + z$, have a simultaneous solution, namely $(1, 1, -1)$, f is degenerate. Hence we use the formula in Equation 4.5 to account for degeneracy. We compute the local zeta function using SPF and show that a resulting pole in the zeta function arises from the sum over the degenerate faces of $\Gamma(f)$.

Since f is the product of linear polynomials, N_1 consists of the sets

$$\{(a, a, b)\} \cup \{(a, b, -a)\} \cup \{(a, b, -b)\}$$

with $a, b \in \mathbb{F}_p$. Thus counting the cardinalities of these sets and taking out intersections, we have that $|N_1| = 3p^2 - 2p$. In addition, we have the following congruences for the partial derivatives of f:

$$\frac{\partial f}{\partial x} \equiv (x + z)(y + z) + (x - y)(y + z) \equiv 0 \pmod p$$
$$\frac{\partial f}{\partial y} \equiv (-1)(x + z)(y + z) + (x - y)(x + z) \equiv 0 \pmod p$$

$$\frac{\partial f}{\partial z} \equiv (x-y)(y+z)+(x-y)(x+z) \equiv 0 \pmod p$$

From these congruences, we see that the singular points modulo p are $\{(a,a,-a)\}$ with $a \in \mathbb{F}_p$ so that $|S| = p$. Thus SPF gives:

$$Z(s) = p^{-3}(p^3-3p^2+2p)+\frac{(3p^2-3p)p^{-3}t(1-p^{-1})}{1-p^{-1}t}+\int_{S+p\mathbb{Z}_p^3}|(x-y)(x+z)(y+z)|^s dxdydz$$

$$= p^{-3}(p^3-3p^2+2p)+\frac{(3p^2-3p)p^{-3}t(1-p^{-1})}{1-p^{-1}t}+\sum_{a\in\mathbb{F}_p}\int_{(a,a,-a)+p\mathbb{Z}_p^3}|(x-y)(x+z)(y+z)|^s dxdydz$$

$$= p^{-3}(p^3-3p^2+2p)+\frac{(3p^2-3p)p^{-3}t(1-p^{-1})}{1-p^{-1}t}+\sum_{a\in\mathbb{F}_p}p^{-3}\int_{\mathbb{Z}_p^3}|(px-py)(px+pz)(py+pz)|^s dxdydz$$

$$= p^{-3}(p^3-3p^2+2p)+\frac{(3p^2-3p)p^{-3}t(1-p^{-1})}{1-p^{-1}t}+\sum_{a\in\mathbb{F}_p}p^{-3}t^3\int_{\mathbb{Z}_p^3}|(x-y)(x+z)(y+z)|^s dxdydz$$

Since this last integral is $Z(s)$, and it is independent of the value a, we may subtract it from the left side to obtain:

$$Z(s) = \frac{(1-p^{-1})(1-2p^{-1}+2p^{-1}t-p^{-2}t)}{(1-p^{-1}t)(1-p^{-2}t^3)}$$

Now we compare this result with the calculation using the formula from Equation 4.5:

$$Z(s) = \int_{\mathbb{Z}_p^3}|(x-y)(x+z)(y+z)|^s dxdydz$$

$$= \sum_{k\in\mathbb{N}^3}\int_{p^{k_1}\mathbb{Z}_p^*\times p^{k_2}\mathbb{Z}_p^*\times p^{k_3}\mathbb{Z}_p^*}|(x-y)(x+z)(y+z)|^s dxdydz$$

$$= \sum_{\tau}\sum_{k\in\mathbb{N}^3\cap\Delta_\tau}\int_{p^{k_1}\mathbb{Z}_p^*\times p^{k_2}\mathbb{Z}_p^*\times p^{k_3}\mathbb{Z}_p^*}|(x-y)(x+z)(y+z)|^s dxdydz$$

Separating the summands for τ when f_τ is non-degenerate, we are left with sums over two cones which are associated to the two degenerate faces of $\Gamma(f)$. We compute the non-degenerate face information using the computer program Polygusa which runs in conjunction with the source code provided by Fukuda. In particular, we list the supporting hyperplanes given by Polygusa and find values for L_τ and S_{Δ_τ} for the non-denegerate faces. We exclude the cases where $p = 2, 3$ since there are extra degenerate faces for these primes. Here we let $v_1 = p^{-3}((p-1)^3 - \frac{p(p-1)^2(p^s-1)}{p^{s+1}-1})$ and $v_2 = p^{-3}(p-1)^3$.[5],[2]

τ_i	supp. hyperplane of τ_i	Δ_{τ_i}	f_{τ_i}	L_{τ_i}	$S_{\Delta_{\tau_i}}$
τ_1	$y=0$	$\{\lambda(0,1,0)\}$	x^2z+xz^2	v_1	$\frac{1}{p-1}$
τ_2	$x+y-1=0$	$\{\lambda(1,1,0)\}$	x^2z-yz^2	v_1	$\frac{1}{p^{2+s}-1}$
τ_3	$x=0$	$\{\lambda(1,0,0)\}$	$-y^2z-yz^2$	v_1	$\frac{1}{p-1}$
τ_4	$x+z-1=0$	$\{\lambda(1,0,1)\}$	$-y^2z-xy^2$	v_1	$\frac{1}{p^{2+s}-1}$
τ_5	$z=0$	$\{\lambda(0,0,1)\}$	x^2y-xy^2	v_1	$\frac{1}{p-1}$
τ_6	$y+z-1=0$	$\{\lambda(0,1,1)\}$	$-xy^2+x^2z$	v_1	$\frac{1}{p^{2+s}-1}$
τ_7	degenerate				
τ_8	$x=1, y=0$	$\{\lambda_1(0,1,0)+\lambda_2(1,1,0)\}$	$-xy^2$	v_2	$\frac{1}{(p-1)(p^{2+s}-1)}$
τ_9	$x=0, y=1$	$\{\lambda_1(1,1,0)+\lambda_2(1,0,0)\}$	$-yz^2$	v_2	$\frac{1}{(p-1)(p^{2+s}-1)}$
τ_{10}	$z=-y+3, x=0$	$\{\lambda_1(1,0,0)+\lambda_2(1,1,1)\}$	$-y^2z-yz^2$	v_1	$\frac{1}{(p-1)(p^{3+3s}-1)}$
τ_{11}	$x=0, z=0$	$\{\lambda_1(1,0,0)+\lambda_2(1,0,1)\}$	$-y^2z$	v_2	$\frac{1}{(p-1)(p^{2+s}-1)}$
τ_{12}	$x=2, z=0$	$\{\lambda_1(1,0,1)+\lambda_2(0,0,1)\}$	$-xy^2$	v_2	$\frac{1}{(p-1)(p^{2+s}-1)}$
τ_{13}	$y=-x+3, z=0$	$\{\lambda_1(0,0,1)+\lambda_2(1,1,1)\}$	x^2y-xy^2	v_1	$\frac{1}{(p-1)(p^{3+3s}-1)}$
τ_{14}	$y=1, z=0$	$\{\lambda_1(0,0,1)+\lambda_2(0,1,1)\}$	x^2y	v_2	$\frac{1}{(p-1)(p^{2+s}-1)}$
τ_{15}	$y=0, z=1$	$\{\lambda_1(0,1,1)+\lambda_2(0,1,0)\}$	x^2z	v_2	$\frac{1}{(p^{2+s}-1)(p-1)}$
τ_{16}	$z=-x+3, y=0$	$\{\lambda_1(0,1,0)+\lambda_2(1,1,1)\}$	x^2z+xz^2	v_1	$\frac{1}{(p-1)(p^{3+3s}-1)}$
τ_{17}	$(1,0,2)$	$\{\lambda_1(0,1,0)+\lambda_2(1,1,0)+\lambda_3(1,1,1)\}$	xz^2	v_2	$\frac{1}{(p-1)(p^{2+s}-1)(p^{3+3s}-1)}$
τ_{18}	$(0,1,2)$	$\{\lambda_1(1,1,0)+\lambda_2(1,0,0)+\lambda_3(1,1,1)\}$	$-yz^2$	v_2	$\frac{1}{(p-1)(p^{2+s}-1)(p^{3+3s}-1)}$
τ_{19}	$(0,2,1)$	$\{\lambda_1(1,0,0)+\lambda_2(1,0,1)+\lambda_3(1,1,1)\}$	$-y^2z$	v_2	$\frac{1}{(p-1)(p^{2+s}-1)(p^{3+3s}-1)}$
τ_{20}	$(1,2,0)$	$\{\lambda_1(1,0,1)+\lambda_2(0,0,1)+\lambda_3(1,1,1)\}$	$-xy^2$	v_2	$\frac{1}{(p-1)(p^{2+s}-1)(p^{3+3s}-1)}$
τ_{21}	$(2,1,0)$	$\{\lambda_1(0,0,1)+\lambda_2(0,1,1)+\lambda_3(1,1,1)\}$	x^2y	v_2	$\frac{1}{(p-1)(p^{2+s}-1)(p^{3+3s}-1)}$
τ_{22}	$(2,0,1)$	$\{\lambda_1(0,1,0)+\lambda_2(0,1,1)+\lambda_3(1,1,1)\}$	x^2z	v_2	$\frac{1}{(p-1)(p^{2+s}-1)(p^{3+3s}-1)}$
τ_{23}	$(1-t)(2,0,1)+t(2,1,0)$	$\{\lambda_1(0,1,1)+\lambda_2(1,1,1)\}$	x^2y+x^2z	v_1	$\frac{1}{(p^{2+s}-1)(p^{3+3s}-1)}$
τ_{24}	$(1-t)(1,0,2)+t(0,1,2)$	$\{\lambda_1(1,1,0)+\lambda_2(1,1,1)\}$	xz^2-yz^2	v_1	$\frac{1}{(p^{2+s}-1)(p^{3+3s}-1)}$
τ_{25}	$(1-t)(0,2,1)+t(1,2,0)$	$\{\lambda_1(1,0,1)+\lambda_2(1,1,1)\}$	$-y^2z-xy^2$	v_1	$\frac{1}{(p^{2+s}-1)(p^{3+3s}-1)}$

Now we are left with two sums which are associated to the two degenerate faces of $\Gamma(f)$. We denote these faces τ_1 and τ_2 and compute their corresponding sums. The polynomials f_{τ_1} and f_{τ_2} are both equal to f since these faces contain all support points. Their respective cones are

$$\Delta_{\tau_1} = \{\lambda(1,1,1) \mid \lambda > 0\}$$

and

$$\Delta_{\tau_2} = \{(0,0)\}.$$

Thus we may use this information to compute the following sums:

$$I_{\tau_1} := \sum_{k \in \mathbb{N}^3 \cap \Delta_{\tau_1}} \int_{p^{k_1}\mathbb{Z}_p^* \times p^{k_2}\mathbb{Z}_p^* \times p^{k_3}\mathbb{Z}_p^*} |(x-y)(x+z)(y+z)|^s dxdydz$$

and

$$I_{\tau_2} := \sum_{k \in \mathbb{N}^3 \cap \Delta_{\tau_2}} \int_{p^{k_1}\mathbb{Z}_p^* \times p^{k_2}\mathbb{Z}_p^* \times p^{k_3}\mathbb{Z}_p^*} |(x-y)(x+z)(y+z)|^s dxdydz.$$

For I_{τ_1}, we integrate over the cone $\Delta_{\tau_1} = \{(\lambda,\lambda,\lambda) \mid \lambda > 0\}$. Thus the sum becomes:

$$I_{\tau_1} = \sum_{\lambda=1}^{\infty} \int_{p^{\lambda}\mathbb{Z}_p^* \times p^{\lambda}\mathbb{Z}_p^* \times p^{\lambda}\mathbb{Z}_p^*} |(x-y)(x+z)(y+z)|^s dxdydz$$

Making the change of variables $x = p^{\lambda}u_1, y = p^{\lambda}u_2, z = p^{\lambda}u_3$, the change in measure is $dxdydz = p^{-3}u_1u_2u_3$ and we see that:

$$\begin{aligned} I_{\tau_1} &= \sum_{\lambda=1}^{\infty} p^{-3} \int_{(\mathbb{Z}_p^*)^3} |(p^{\lambda}u_1 - p^{\lambda}u_2)(p^{\lambda}u_1 + p^{\lambda}u_3)(p^{\lambda}u_2 + p^{\lambda}u_3)|^s du_1du_2du_3 \\ &= \sum_{\lambda=1}^{\infty} p^{-3}t^{3\lambda} \int_{(\mathbb{Z}_p^*)^3} |(u_1-u_2)(u_1+u_3)(u_2+u_3)|^s du_1du_2du_3 \\ (1) &= \frac{p^{-3}t^3}{1-p^{-3}t^3} I_1 \end{aligned}$$

where

$$I_1 = \int_{(\mathbb{Z}_p^*)^3} |(u_1-u_2)(u_1+u_3)(u_2+u_3)|^s du_1du_2du_3.$$

By SPF we have that

$$I_1 = p^{-3}((p-1)^3 - 3(p-1)^2 + 2(p-1)) + \frac{(3(p-1)^2 - 3(p-1))p^{-3}t(1-p^{-1})}{1-p^{-1}t}$$

$$+\sum_{a\in\mathbb{F}_p^*}\int_{(a,a,-a)+p\mathbb{Z}_p^3}|(u_1-u_2)(u_1+u_3)(u_2+u_3)|^s du_1du_2du_3$$

Making the change of variables $u_1 = a+px_1, u_2 = a+py_1, u_3 = -a+pz_1$, for the last integral we obtain the following after simplification:

$$I_1 = p^{-3}((p-1)^3-3(p-1)^2+2(p-1))+\frac{(3(p-1)^2-3(p-1))p^{-3}t(1-p^{-1})}{1-p^{-1}t}$$

$$+\sum_{a\in\mathbb{F}_p^*}p^{-3}\int_{\mathbb{Z}_p^3}|(px_1-py_1)(px_1+pz_1)(py_1+pz_1)|^s dx_1dy_1dz_1,$$

and hence that

$$I_1 = p^{-3}((p-1)^3-3(p-1)^2+2(p-1))+\frac{(3(p-1)^2-3(p-1))p^{-3}t(1-p^{-1})}{1-p^{-1}t}$$

$$+\sum_{a\in\mathbb{F}_p^*}p^{-3}t^3\int_{\mathbb{Z}_p^3}|(x_1-y_1)(x_1+z_1)(y_1+z_1)|^s dx_1dy_1dz_1$$

or recognizing the last integral as $Z(s)$ we have that

$$I_1 = p^{-3}((p-1)^3-3(p-1)^2+2(p-1))+\frac{(3(p-1)^2-3(p-1))p^{-3}t(1-p^{-1})}{1-p^{-1}t}+(p-1)p^{-3}t^3Z(s)$$

Now from equation (1), we have that:

$$I_{\tau_1} = \frac{p^{-3}t^3}{1-p^{-3}t}I_1$$

or that

$$I_{\tau_1} = \frac{(p^{-3}t^3)}{1-p^{-3}t^3}[L+(p-1)p^{-3}t^3Z(s)]$$

where

$$L = p^{-3}((p-1)^3-3(p-1)^2+2(p-1))+\frac{(3(p-1)^2-3(p-1))p^{-3}t(1-p^{-1})}{1-p^{-1}t}.$$

Thus we have computed I_{τ_1} in terms of $Z(s)$. We consider I_{τ_2} and note that for I_{τ_2}, the only point summed in Δ_{τ_2} is $(0,0)$ since this is the entire cone associated with the improper face. Thus $I_{\tau_2} = I_1$.

We now combine all summands of the original partition into cones over $\mathbb{N}^3$ together to obtain the following sum:

$$Z(s) = \sum_{\tau \notin \{\tau_1, \tau_2\}} S_{\Delta_\tau} L_\tau + L + (p-1)p^{-3}t^3 Z(s) + \frac{p^{-3}t^3 L}{1 - p^{-3}t^3} + \frac{p^{-3}t^3 (p-1)p^{-3}t^3 Z(s)}{1 - p^{-3}t^3}$$

so combining all terms which are multiples of the original zeta function $Z(s)$ and moving them to the left side of the previous equation, we have:

$$Z(s) - (p-1)p^{-3}t^3 Z(s) - \frac{(p-1)p^{-6}t^6 Z(s)}{1 - p^{-3}t^3} = \sum_{\tau \notin \{\tau_1, \tau_2\}} S_{\Delta_\tau} L_\tau + L + \frac{p^{-3}t^3 L}{1 - p^{-3}t^3}$$

and hence

$$Z(s)[1 - (p-1)p^{-3}t^3 - \frac{(p-1)p^{-6}t^6}{1 - p^{-3}t^3}] = \sum_{\tau \notin \{\tau_1, \tau_2\}} S_{\Delta_\tau} L_\tau + L + \frac{p^{-3}t^3 L}{1 - p^{-3}t^3}$$

$$Z(s)[\frac{1 - (p-1)p^{-3}t^3)(1 - p^{-3}t^3) - (p-1)p^{-6}t^6}{1 - p^{-3}t^3}] = \sum_{\tau \notin \{\tau_1, \tau_2\}} S_{\Delta_\tau} L_\tau + L + \frac{p^{-3}t^3 L}{1 - p^{-3}t^3}$$

$$Z(s)[\frac{1 - p^{-2}t^3}{1 - p^{-3}t^3}] = \sum_{\tau \notin \{\tau_1, \tau_2\}} S_{\Delta_\tau} L_\tau + L + \frac{p^{-3}t^3 L}{1 - p^{-3}t^3}$$

so we see that a pole is contributed by the combined sum of the singular integrals corresponding to the degenerate faces of $\Gamma(f)$. In order to see that this pole does not come from the other terms of the sum $\sum_{\tau \notin \{\tau_1, \tau_2\}} S_{\Delta_\tau} L_\tau$, we compute the denominators of the S_{Δ_τ} so show that no such pole can come from these denominators. Below we give these candidate poles explicitly which arise from each facet of $\Gamma(f)$.

Normals to the i^{th} facet	$p^{\sigma(a_i)+m(a_i)s} - 1$
a_1	$p^{2+s} - 1$
a_2	$p - 1$
a_3	$p^{3+3s} - 1$
a_4	$p^{2+s} - 1$
a_5	$p - 1$
a_6	$p^{2+s} - 1$
a_7	$p - 1$

Thus since no candidate pole yields the same pole resulting from the denominator $1 - p^{-2}t^3$, this pole arises from the degenerate faces and is a new pole of the form given in Proposition 4.8.

6. When Degeneracy Contributes to the Order of an Existing Pole

Now we want to consider the case when the singular points do not satisfy Condition 2 stated earlier. Here we compute the local zeta function of a polynomial with three homogeneous linear factors in three variables.

Example 6.1. Let $f(x,y,z) = (x-y)(x+y+z)(x-y+z)$. Then $N_1 = \{(a,a,b)\} \cup \{(a,b,-a-b)\} \cup \{(a,b,-a+b)\}$ with $a,b \in \mathbb{F}_p$, and $S = \{(a,a,-2a)\} \cup \{(a,0,-a)\} \cup \{(a,a,0)\}$ so $|N_1| = 3p^2 - 3p + 1$ and $|S| = 3p - 2$. Thus we have by SPF:

$$Z(s) = \int_{\mathbb{Z}_p^3} |(x-y)(x+y+z)(x-y+z)|^s dxdydz$$

$$= p^{-3}(p^3 - 3p^2 + 3p - 1) + \frac{(3p^2 - 6p - 3)p^{-3}t(1-p^{-1})}{1 - p^{-1}t} + p^{-3}t^3 Z(s)$$

$$+ \sum_{a \in \mathbb{F}_p^*} \int_{(a,a,-2a)+p\mathbb{Z}_p^3} |(x-y)(x+y+z)(x-y+z)|^s dxdydz$$

$$+ \sum_{a \in \mathbb{F}_p^*} \int_{(a,a,0)+p\mathbb{Z}_p^3} |(x-y)(x+y+z)(x-y+z)|^s dxdydz$$

$$+ \sum_{a \in \mathbb{F}_p^*} \int_{(a,0,-a)+p\mathbb{Z}_p^3} |(x-y)(x+y+z)(x-y+z)|^s dxdydz$$

Now we evaluate the integral

$$I_1 := \int_{(a,a,-2a)+p\mathbb{Z}_p^3} |(x-y)(x+y+z)(x-y+z)|^s dxdydz.$$

Making the change of variables, $x = a + px_1, y = a + py_1, z = -2a + pz_1$, the change in measure is $dxdydz = p^{-3}dx_1dy_1dz_1$ and I_1 becomes:

$$I_1 = p^{-3} \int_{\mathbb{Z}_p^3} |(px_1 - py_1)(px_1 + py_1 + pz_1)(-2a + p(x_1 - y_1 + z_1))|^s dx_1dy_1dz_1$$

$$= p^{-3}t^2 \int_{\mathbb{Z}_p^3} |(x_1 - y_1)(x_1 + y_1 + z_1)(-2a + p(x_1 - y_1 + z_1))|^s dx_1dy_1dz_1$$

so since $a \in \mathbb{F}_p^*$, $-2a + p(x_1 - y_1 + z_1)$ is a unit for any values of x_1, y_1, and z_1. Thus we have

$$I_1 =: p^{-3}t^2 Z_1(s) = p^{-3}t^2 \int_{\mathbb{Z}_p^3} |(x_1 - y_1)(x_1 + y_1 + z_1)|^s dx_1dy_1dz_1$$

$$= p^{-3}t^2[p^{-3}(p^3-2p^2+p)+\frac{p^{-3}t(2p^2-2p)(1-p^{-1})}{1-p^{-1}t}+p^2t^2Z_1(s)]$$

using SPF since the singular points now satisfy Condition 2. Moving all multiples of $Z_1(s)$ to the left side and solving for $Z_1(s)$, we have

$$Z_1(s)=\frac{(1-p^{-1})^2}{(1-p^{-1}t)^2}.$$

Thus

$$I_1=\frac{p^{-3}t^2(1-p^{-1}t)^2}{(1-p^{-1}t)^2}.$$

It is easily verified that the two other integrals $\int_{(a,a,0)+p\mathbb{Z}_p^3}|(x-y)(x+y+z)(x-y+z)|^s dxdydz$ and $\int_{(a,0,-a)+p\mathbb{Z}_p^3}|(x-y)(x+y+z)(x-y+z)|^s dxdydz$ are equal to I_1, so we have the following for $Z(s)$:

$$Z(s)=\frac{(1-p^{-1})^3)(1-p^{-1}t)^2}{(1-p^{-1}t)^2}+\frac{3p^{-1}t(1-p^{-1}t)(1-p^{-1})^3}{(1-p^{-1}t)^2}$$
$$+p^{-3}t^3Z(s)+3(p-1)p^{-3}t^2[\frac{(1-p^{-1})^2}{(1-p^{-1}t)^2}]$$

so again moving all multiples of $Z(s)$ to the left and solving for $Z(s)$, we have

$$Z(s)=\frac{(1-p^{-1})^3[(1-p^{-1}t)^2+3p^{-1}t(1-p^{-1}t)+3p^{-2}t^2]}{(1-p^{-1}t)^2(1-p^{-3}t^3)}$$
$$=\frac{(1-p^{-1})^3[1+p^{-1}t+p^{-2}t^2]}{(1-p^{-1}t)^3(1+p^{-1}t+p^{-2}t^2)}$$
$$=\frac{(1-p^{-1})^3}{(1-p^{-1}t)^3}$$

Now we want to examine how the poles of the local zeta function compare to those given by the sums over the non-degenerate faces in Equation 4.5. To do this we again examine the candidate poles potentially given by the non-degenerate faces. We list the facets of $\Gamma(f)$ below:

Supporting hyperplane of the facet τ	Spanning vector for Δ_τ	$p^{\sigma(a_i)+m(a_i)s}-1$
$z=0$	$(0,0,1)$	$p-1$
$x+y-1=0$	$(1,1,0)$	$p^{2+s}-1$
$x=0$	$(1,0,0)$	$p-1$
$y=0$	$(0,1,0)$	$p-1$
$x+y+z-3=0$	$(1,1,1)$	$p^{3+3s}-1$

So we see that the pole of order 3 at $s_0 = -1$ comes from one of the candidate poles given in the S_{Δ_τ} of the Newton polyhedron method as well as the sum of the degenerate faces. So in conclusion, the order of this pole results from the degeneracy since the non-degenerate faces give this pole with maximal order 2.

7. Conclusion

We focused on the study of degenerate polynomials and the poles that arise from summing over cones of degenerate faces, however there is little that is known in general about the local zeta functions of such polynomials. In order to focus on a particular class of such polynomials, we chose those which were degenerate for all primes p. We were able to appeal to the geometry of their Newton polyhedra in order to study when new poles arose from the degeneracy, but even in this restricted case, there are still questions regarding what the local zeta functions of these polynomials are in general. Studying larger classes of polynomials might give further insight into how degeneracy interacts with candidate poles and how numerators of the local zeta functions change if no new poles result from the degeneracy.

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