

Degenerate Polynomials and Poles

A Look into Degeneracy

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A Brief Introduction to p -adic Numbers

Definition 0.1 Define the p -adic valuation on \mathbb{Q} to be the function $v_p : \mathbb{Q} \longrightarrow \mathbb{Z} \setminus \{0\}$ such that if $x \in \mathbb{Q}$, $x = p^{v_p(x)} x'$, where $p \nmid x'$.

Definition 0.2 For $x \in \mathbb{Q}$, the p -adic absolute value is defined as follows:

$$|x|_p = \begin{cases} p^{-v_p(x)} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

In order to obtain the p -adic numbers, denoted \mathbb{Q}_p , we must complete \mathbb{Q} with respect to the p -adic absolute value. We then extend the absolute value to \mathbb{Q}_p .

The Ring of p -adic Integers

The set denoted $\mathbb{Z}_p := \{x \in \mathbb{Q}_p \mid |x| \leq 1\}$ is a subring of \mathbb{Q}_p referred to as the p -adic integers.

A useful way to represent a p -adic integer $x \in \mathbb{Z}_p$ is by an expansion in powers of p as follows:

$$x = a_0 + a_1p + a_2p^2 + \dots$$

where $0 \leq a_i \leq p - 1$ for all i . We denote by \mathbb{Z}_p^* the units of \mathbb{Z}_p , that is, those elements for which $a_0 \neq 0$, where a_0 is the first element of its expansion.

Igusa's local zeta function

We define Igusa's local zeta function to be:

$$Z(s) = \int_{\mathbb{Z}_p^n} |f(x_1, \dots, x_n)|^s dx_1 \dots dx_n,$$

where $s \in \mathbb{C}$ and $\operatorname{Re}(s) > 0$. We also adopt the convention that $t = p^{-s}$.

Ways to Compute Local Zeta functions:

- 1) Stationary Phase Formula (Igusa, 1994)
- 2) Newton Polyhedron Method

The Stationary Phase Formula

Let

$$N_1 = \{x \in \mathbb{F}_p^n \mid f(x) \equiv 0 \pmod{p}\}$$

and

$$S = \{y \in N_1 \mid \frac{\partial f}{\partial x_i}(y) \equiv 0 \pmod{p}, 1 \leq i \leq n\}.$$

Then

$$Z(s) = p^{-n}(p^n - |N_1|) + \frac{(|N_1| - |S|)p^{-n}t(1 - p^{-1})}{1 - p^{-1}t} + \sum_{s \in S} \int_{s+p\mathbb{Z}_p^n} |f(x)|^s dx$$

Newton polyhedra

Definition 0.3 *Let*

$$f = \sum_{k=(k_1, \dots, k_n) \in \mathbb{N}^n} a_k x_1^{k_1} \cdots x_n^{k_n} \in \mathbb{Z}[x_1, \dots, x_n]$$

with $f(0) = 0$, and let \mathbb{R}^+ be the set of all nonnegative real numbers. Define the support of f to be $\text{supp}(f) = \{k \in \mathbb{N}^n \mid a_k \neq 0\}$, and let $\Gamma'(f)$ denote the convex hull of $\text{supp}(f)$. Then the Newton polyhedron denoted $\Gamma(f)$ is defined by the following equation

$$\Gamma(f) = \Gamma'(f) + (\mathbb{R}^+)^n.$$

Newton Polyhedra cont'd

Definition 0.4 *A face of the Newton polyhedron is the intersection of a supporting hyperplane and the Newton polyhedron $\Gamma(f)$. We call τ a facet if the codimension of τ is 1.*

Definition 0.5 *Let*

$$f = \sum_{k=(k_1, \dots, k_n) \in \mathbb{N}^n} a_k x_1^{k_1} \cdots x_n^{k_n} \in \mathbb{Z}[x_1, \dots, x_n]$$

with $f(0) = 0$. Then we define:

$$m(a) := \inf_{x \in \Gamma(f)} \{a \cdot x\}$$

More about Newton Polyhedra

Definition 0.6 *The first meet locus of a is defined as:*

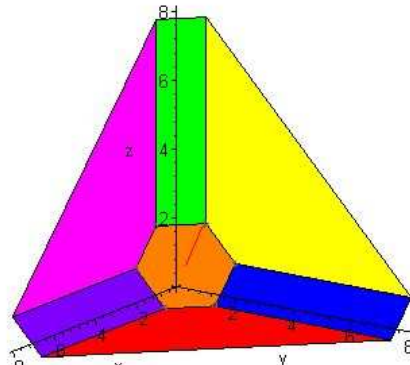
$$F(a) = \{x \in \Gamma(f) \mid x \cdot a = m(a)\}$$

Definition 0.7 *Let τ be a face of $\Gamma(f)$. Then the cone associated to τ is:*

$$\Delta_\tau = \{a \in (\mathbb{R}^+)^n \mid F(a) = \tau\}.$$

Example:

Newton polyhedron



Using the Newton Polyhedron

Definition 0.8 We define f_τ to be the part of the polynomial f with support equal to $\text{supp}(f) \cap \tau$.

Definition 0.9 A polynomial $f(x_1, \dots, x_n) \in \mathbb{Z}[x_1, \dots, x_n]$ is nondegenerate with respect to its Newton polygon if for every face τ , the system of congruences:

$$f_\tau \equiv 0 \pmod{p}$$

$$\frac{\partial f_\tau}{\partial x_i} \equiv 0 \pmod{p}$$

has no solution in $(\mathbb{F}_p^*)^n$.

Using the Newton Polyhedron cont'd

Assuming f is a polynomial which is non-degenerate, the local zeta function is:

$$Z(s) = \sum_{\tau \in \Gamma(f)} S_{\Delta_\tau} L_\tau$$

with

$$S_{\Delta_\tau} = \sum_{k \in \mathbb{N}^n \cap \Delta_\tau} p^{-\sigma(k) - m(k)s},$$

$$L_\tau = p^{-n} \left((p-1)^n - \frac{|N_\tau| p(p^s - 1)}{p^{s+1} - 1} \right)$$

where

$$N_\tau = \{x \in (\mathbb{F}_p^*)^n \mid f_\tau(x) \equiv 0 \pmod{p}\}$$

and $\sigma(k)$ is the sum of the components of k .

Constructing Degenerate Polynomials

Consider $f = \prod_{i=1}^m a_i$ where each a_i is a linear homogeneous polynomial in n variables. If two of these factors have a simultaneous nonzero solution mod p , then f is degenerate.

Idea for why this is true: product rule

New Pole from Degeneracy

Consider the polynomial $f(x, y, z) = (x - y)(x + z)(y + z)$. The first two factors have a simultaneous solution mod p , namely $(1, 1, -1)$, so f is degenerate.

Using SPF, we calculate the local zeta function for f and obtain the following:

$$Z(s) = \frac{(1 - p^{-1})[1 - 2p^{-1} + 2p^{-1}t - p^{-2}t]}{(1 - p^{-1}t)(1 - p^{-2}t^3)}$$

Why is This A New Pole?

To see why this pole does not come from the non-degenerate integrals used in the Newton polyhedron method, we look at the facets since the poles of f come from the denominators of each S_{Δ_τ} which are related to the normals to the facets of $\Gamma(f)$

Candidate Poles

Normals to the i^{th} facet	$p^{\sigma(a_i)+m(a_i)s} - 1$
a_1	$p^{2+s} - 1$
a_2	$p - 1$
a_3	$p^{3+3s} - 1$
a_4	$p^{2+s} - 1$
a_5	$p - 1$
a_6	$p^{2+s} - 1$
a_7	$p - 1$

A Pole from Degeneracy

So since none of these candidate poles correspond to the new pole, it must result from the degenerate integrals. Specifically in our case, the pole results from the sum of two degenerate integrals. In general, whenever all singular points satisfy all linear factors in the product of f , a pole always results from the degeneracy of f .

Further Research

When the singular points do not satisfy the previous condition, new poles do not always come from the degeneracy of f . One example where this can be seen is with the polynomial $f(x, y, z) = (x - y)(x + y + z)(x - y + z)$. Thus it may be possible that new poles do not result from degeneracy unless certain conditions are placed on the singular points.

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