CASE I_0^*

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Abstract. The 1999 REU computed the Igusa Zeta function for the case I_0*. Since there methods completely generalize to an Igusa Zeta function in an arbitrary field extension, the relation between the Igusa Zeta function and its Poincaré series allows us to compute explicity the $|N_{d,e}|$. Using these, we can then compute the general Weil Zeta Function and Weil Poincaré series, as well as attempt to combine the Igusa and Weil Poincaré series.

1. Computing $|N_{d,e}|$

The 1999 REU computed the Igusa Zeta function of an elliptic curve in the I_0^* case to be

$$Z_I(t) = (1-p^{-1}) + (1-p^{-1})p^{-1}t\left(\frac{1-p^{-1}}{1-p^{-1}t}\right) + p^{-2}t^2(1-p^{-1})$$

$$+p^{-3}t^3\left((1-Mp^{-1}) + Mp^{-1}t\left(\frac{1-p^{-1}}{1-p^{-1}t}\right)\right),$$

where M depends on the number of roots of a cubic polynomial in $\mathbb{F}_p$.

1.1. The General Igusa Zeta Function.

The Igusa Zeta Function over the ideals of $\mathbb{Z}_p$ was calculated using SPF, a method that completely generalizes to ideals of rings of higher dimension. In particular, if we compute the Igusa Zeta Function in a vector space of dimension d over $\mathbb{Z}_p$, and let $q = p^d$, we have

$$Z_I(t) = (1-q^{-1}) + (1-q^{-1})q^{-1}t\left(\frac{1-q^{-1}}{1-q^{-1}t}\right) + q^{-2}t^2(1-q^{-1})$$

$$+q^{-3}t^3\left((1-Mq^{-1}) + Mq^{-1}t\left(\frac{1-q^{-1}}{1-q^{-1}t}\right)\right),$$

where M now depends on the number of roots of the cubic polynomial in $\mathbb{F}_q$.

1.2. **Computing M.** M is the number of zeroes of the polynomial $f(x) = x^3 - a_{2,1}x^2 - a_{4,2}x - a_{6,3}$ in $\mathbb{F}_q$. In the case I_0^*, we are assuming that $f(x)$ has three distinct roots in $\overline{\mathbb{F}_p}$. There are four possibilities for this polynomial.

(1) $f(x)$ is irreducible over $\mathbb{F}_p$, with splitting field of degree 6.
(2) $f(x)$ is irreducible over $\mathbb{F}_p$, with splitting field of degree 3.
(3) $f(x)$ has exactly one root in $\mathbb{F}_p$.
(4) $f(x)$ factors into three linear terms over $\mathbb{F}_p$.

The behavior of $f(x)$ in $\mathbb{F}_p$ completetly characterizes M in $\mathbb{F}_q$, since every extension of $\mathbb{F}_p$ of degree d is isomorphic. In the 4 cases listed above, we have

(1) If $3|d$ but $6 \nmid d$, then $\mathbb{F}_q$ contains exactly one x_0 such that $f(x_0) = 0$. Therefore $M = 1$. If $6|d$, then $\mathbb{F}_q$ contains three x_0 such that $f(x_0) = 0$, so $M = 3$. If $3 \nmid d$, $\mathbb{F}_q$ does not contain any roots of $f(x)$, so $M = 0$.
(2) If $3|d$, then $M = 3$, and if $3 \nmid d$, then $M = 0$.
(3) If $2|d$, then $M = 3$. If $2 \nmid d$, then $M = 1$.
(4) Regardless of d, $M = 3$.

1.3. **The Poincaré Series and $|N_{d,e}|$.** Using the relation between the Poincaré Series and the Zeta Function

$$P(t) = \frac{1 - tZ(t)}{1 - t} \ ,$$

we obtain

$$P(t) = \frac{t^4M - t^4 + t^2q^2 - t^2q + q^4}{q^3(q - t)} \ .$$

We also know that $P(T) = \sum_{e=0}^{\infty} |N_{d,e}|q^{-2e}T^e$. We can simplify this sum by writing $P(Tq^2) = \sum_{e=0}^{\infty} |N_{d,e}|T^e$. The fraction representation gives $P(Tq^2) = \frac{(q^4M - q^4)T^4 + (q^2 - q)T^2 + 1}{1 - qT}$. Since these two must be identically equal, we have

$$\begin{aligned}
\frac{(q^4M - q^4)T^4 + (q^2 - q)T^2 + 1}{1 - qT} &= \sum_{e=0}^{\infty} |N_{d,e}|T^e \\
(q^4M - q^4)T^4 + (q^2 - q)T^2 + 1 &= \sum_{e=0}^{\infty} |N_{d,e}|T^e - \sum_{e=0}^{\infty} |N_{d,e}|qT^{e+1} \\
&= |N_{d,0}| + \sum_{e=1}^{\infty} (|N_{d,e}| - q|N_{d,e-1}|)T^e
\end{aligned}$$

This forces several recursion relations among the $|N_{d,e}|$.

(1) $|N_{d,0}| = 1$

(2) $|N_{d,1}| - q|N_{d,0}| = |N_{d,1}| - q = 0$. Therefore $|N_{d,1}| = q$.
(3) $|N_{d,2}| - q|N_{d,1}| = |N_{d,2}| - q^2 = q^2 - q$. Therefore $|N_{d,2}| = 2q^2 - q$.
(4) $|N_{d,3}| - q|N_{d,2}| = |N_{d,3}| - 2q^3 + q^2 = 0$. Therefore $|N_{d,3}| = 2q^3 - q^2$.
(5) $|N_{d,4}| - q|N_{d,3}| = |N_{d,4}| - 2q^4 + q^3 = q^4M - q^4$. Therefore $|N_{d,4}| = q^4M + q^4 - q^3$.
(6) For $e \geq 5$, $|N_{d,e}| - q|N_{d,e-1}| = 0$, so $|N_{d,e}| = q|N_{d,e-1}|$. Inductively then, $|N_{d,e}| = q^e + Mq^e - q^{e-1}$.

With this and the information about M, we now know explicity every $|N_{d,e}|$ based on the splitting field of $f(x)$.

2. The Weil Zeta Function and Weil Poincaré Series

Now we can fix e and combine the $|N_{d,e}|$ into the Weil Zeta function $\exp\left(\sum_{d=1}^{\infty} \frac{|N_{d,e}|T^e}{d}\right)$ and the Weil Poincaré series. We know that if $|N_{d,e}| = \alpha_1^d + \alpha_2^d + \cdots + \alpha_j^d - \beta_1^d - \beta_2^d - \cdots - \beta_k^d$, then

$$Z_W(T) = \frac{(1-\beta_1 T)(1-\beta_2 T)\dots(1-\beta_k T)}{(1-\alpha_1 T)(1-\alpha_2 T)\dots(1-\alpha_j T)}.$$

In particular, we have

$$e = 1 \qquad Z_w(T) = \frac{1}{1-pT} \tag{1}$$

$$e = 2 \qquad Z_W(T) = \frac{1-pT}{1-p^2T} \tag{2}$$

$$e = 3 \qquad Z_W(T) = \frac{1-p^2T}{(1-p^3T)^2} \tag{3}$$

For $e \geq 4$, the situation is complicated by the presence of the M term. We begin by splitting the sum into

$$\exp\left(\sum_{d=1}^{\infty} \frac{((p^e)^d - (p^{e-1})^d)T^d}{d}\right) \exp\left(\sum_{d=1}^{\infty} \frac{M(p^e)^d T^d}{d}\right).$$

The first exp can be rewritten in the usual way as $\frac{1-p^{e-1}T}{1-p^eT}$. As noted earlier, M depends on the degree of the splitting field of the polynomial $f(x)$.

2.1. $f(x)$ **Has Splitting Field of Degree 6.** As computed in (1.2), in this case M depends on d mod 6. When $d \equiv 1, 2, 4, 5 \bmod 6, M = 0$, when $d \equiv 3 \bmod 6, M = 1$, and when $d \equiv 0 \bmod 6, M = 3$. Therefore the sum can be rewritten as

$$\exp\left(\sum_{k=1}^{\infty}\frac{(p^e)^{3k}T^{3k}}{3k}\right)\exp\left(\sum_{k=1}^{\infty}\frac{2(p^e)^{6k}T^{6k}}{6k}\right).$$

We must make a brief aside to evaluate this sum. We know that $\sum_{k=1}^{\infty}\frac{x^k}{k} = -\log(1-x)$. Simple substitution shows that $\sum_{k=1}^{\infty}\frac{x^{nk}}{k} = -\log(1-x^n)$. Dividing through by n gives that

$$\sum\frac{x^{nk}}{nk} = -\frac{1}{n}\log(1-x^n).$$

Therefore we have that this expression can be rewritten as

$$\frac{1}{\sqrt[3]{1-(p^eT)^3}}\left(\frac{1}{\sqrt[6]{1-(p^eT)^6}}\right)^2,$$

giving us

$$Z_I(T) = \frac{1-p^{e-1}T}{(1-p^eT)(\sqrt[3]{1-(p^eT)^3})(\sqrt[3]{1-(p^eT)^6})}.$$