POINCARÉ SERIES OF WEIL-IGUSA TYPE FOR ELLIPTIC CURVES

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Abstract. We consider a double Poincaré series associated to a polynomial over $\mathbb{Q}_p$. We see that this function of two variables is a natural way to combine the Weil zeta function and Igusa zeta function. We answer questions regarding the rationality of this Poincaré series, specifically for the case of elliptic curves. This paper extends the work of Diane Meuser in [4] on a related function and relies on the findings of the 1999 Mount Holyoke College REU in [1, 5]. [1]

1. Introduction and Motivation

In this paper we want to count the solutions to elliptic curves in finite p-adic rings. Let $\mathcal{O}_d$ be the the ring of integers in the unramified finite extension of degree d over $\mathbb{Q}_p$. Then we set

$$N_{d,e} = \{(x,y) \in (\mathcal{O}_d/p^e\mathcal{O}_d)^{(2)} | f(x,y) \equiv 0 (\text{mod } p^e)\}.$$

The Weil zeta function is defined by the infinite sum

$$Z_{\text{Weil}}(W) = \exp \sum_{d=1}^{\infty} |N_{d,1}| \frac{W^d}{d}.$$

The Weil zeta function was shown to be rational by Dwork in 1959 using p-adic methods. This rationality implies that there is a recursive relationship among the $|N_{d,1}|$ for sufficiently large d.

The Igusa zeta function is defined by the p-adic integral

$$Z_{\text{Igusa},d}(s) = \int_{\mathcal{O}_d^{(n)}} |f(x)|^s |dx|$$

where dx denotes the Haar measure on K_d normalized so that $\mathcal{O}_d^{(n)}$ has measure 1, and $|\cdot|$ denotes the p-adic absolute value on K_d normalized so that $|x| = p^{-d \operatorname{ord} x}$.

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For a fixed d, the Igusa zeta function is a generating function for the $|N_{d,e}|$'s. This zeta function was proved to be rational by Igusa in 1974 using Hironaka's resolution of singularities. This rationality implies that there is a recursive relationship among the $|N_{d,e}|$ for sufficiently large e.

Both zeta functions have associated Poincaré series defined as

$$P_{\text{Weil}}(W) = \sum_{d=1}^{\infty} |N_{d,1}| W^d \qquad\qquad P_{\text{Igusa},d}(t) = \sum_{e=0}^{\infty} |N_{d,e}| p^{-nde} t^e$$

where $t = p^{-ds}$.

In both cases, the rationality of the zeta function implies the rationality of the associated Poincaré series.

We get the following relationship between the Igusa zeta function and Poincaré series:

$$P_I(t) = \frac{1 - tZ_I(t)}{1-t}. \tag{1}$$

In [1] and [5], the Igusa zeta function was computed for all Kodaira-Néron reduction types of elliptic curves.

We found the following data for all elliptic curves according to the Kodaira-Néron classification as determined by Tate's algorithm: $|N_{d,e}|$, $\mathcal{Z}_{\text{Weil},e}(W)$, $P_{\text{Weil},e}(W)$, $\mathcal{Z}_{\text{Igusa},d}(T)$, $\mathcal{Z}_{\text{Igusa}}(T)$, $P(W,T)$. For more information on the Kodaira-Néron classification of elliptic curves and on elliptic curves in general, see [7, 6].

In [4], Diane Meuser introduced a double Poincaré series which she defined to be

$$P_{Meuser}(T,W) = \sum_{d=1}^{\infty}\sum_{e=0}^{\infty} |N_{d,e}| p^{-2de} T^e W^d.$$

Meuser proved that this series can be rational but in general, is not. In [4], Meuser completely characterizes exactly when $P_{\text{Meuser}}(W,T)$ is a rational function of W and T. For an example of irrationality, in the case of the polynomial $f(x) = x$, we see for all d and e, we have $|N_{d,e}| = 1$. Therefore

$$P_{\text{Meuser}}(W,T) = \sum_{d=1}^{\infty}\sum_{e=0}^{\infty} p^{-de} W^d T^e = \sum_{d=1}^{\infty} W^d \sum_{e=0}^{\infty} (p^{-d}T)^e = \sum_{d=1}^{\infty} \frac{W^d}{1-p^{-d}T}.$$

This cannot be written as the quotient of two polynomials for the reason that $P(W,T)$ has infinity many poles; whenever $T = p^d$ for a positive integer d, we see that $P(W,T)$ blows up.

We modified Meuser's double sum into

$$P(T,W) = \sum_{d=1}^{\infty}\sum_{e=0}^{\infty} |N_{d,e}| p^{-de} T^e W^d.$$

because this gives us a rational function in the case of elliptic curves.

Margaret, this next paragraph needs to be rewritten. We want to convey the idea that $||N_{d,1}| - p^d - 1| \leq 2\sqrt{p^d}$ by a theorem of Hasse. Is this true in general, that we get $||N_{d,e}| - p^{de} - 1|$ is bounded by some lesser order term?

A natural reason for doing this is that one might expect for elliptic curves that $|N_{d,e}| \approx Cp^{de}$ where C is some constant. By multiplying through by p^{-de}, we are normalizing so that $|N_{d,e}|p^{-de} = C + \text{error term}$, where this error term is small when compared with C. We explicitly determine $P(W,T)$ for all elliptic curves according to the Kodaira-Néron classification of elliptic curves as determined by Tate's algorithm. We find that in all cases $P(W,T)$ is a rational function of W and T.

2. Elliptic Curves of Reduction Type I_0

For each elliptic curve in Weierstrass normal form, we consider its reduction modulo p. Although elliptic curves $f(x,y)$ are by definition nonsingular over $\mathbb{Q}_p$, our reduced elliptic curve $\bar{f}(x,y)$ may be singular over $\mathbb{F}_p$. An elliptic curve has reduction type I_0 if $\bar{f}(x,y)$ remains nonsingular modulo p. Each elliptic curve has a discriminant Δ. Being nonsingular modulo p is equivalent to $p \nmid \Delta$. Because Δ is just an integer, reduction type I_0 must occur for all but finitely many primes p.

Suppose $f(x,y)$ is of reduction type I_0. In this case we apply Hensel's Lemma. To begin, suppose $(a,b) \in N_{d,1}$. By the multivariable Hensel's Lemma generalized to $\mathcal{O}_d$, there are $p^{d(e-1)}$ ways to lift (a,b) to $N_{d,e}$. It follows from this that $|N_{d,e}| = p^{d(e-1)}|N_{d,1}|$ whenever $e \geq 1$. Using this information we find that

$$P(W,T) = P_{\text{Weil}}(p^{-1}W)\left(\frac{T}{1-T}\right) + \left(\frac{W}{1-W}\right)$$

.

It follows that $P(W,T)$ is rational by the rationality of the Weil Poincaré series $P_{\text{Weil}}(W)$. For more on the Weil zeta function, see [3].

3. Computing the $|N_{d,e}|$ of an Elliptic Curve with Reduction Type III

The 1999 REU Group at Mount Holyoke College in Number Theory [1] found the Igusa local zeta function for an elliptic curve with reduction type III to have the rational form:

$$Z_I(t) = \frac{1 - p^{-1} - p^{-2}t + p^{-2}t^2 + p^{-3}t - p^{-3}t^2}{1 - p^{-1}t} \tag{2}$$

It is also known that the Igusa Poincaré Series can be expressed in two different forms:

$$P_I(t) = \frac{1 - tZ_I(t)}{1-t} \tag{3}$$

$$P_I(t) = \sum_{e=0}^{\infty} |N_{d,e}| p^{-2e} t^e \tag{4}$$

Since the Kodaira-Néron Reduction Type remains Type III when working in an unramified extension field, K_d, of degree d over $\mathbb{Q}_p$ all of the p's in equations 2 and 4 can be changed to to p^d's. Letting $q = p^d$, over K_d, our new Igusa Local Zeta Function in the extension field of degree d becomes:

$$Z_I(t) = \frac{1 - q^{-1} - q^{-2}t + q^{-2}t^2 + q^{-3}t - q^{-3}t^2}{1 - q^{-1}t} \tag{5}$$

To solve for the $|N_{d,e}|$ plug equation 5 into equation 3 and expand it into an infinite sum. This can be equated with equation 4 the $|N_{d,e}|$ can be solved for. The 1999 REU Group [**?**] found $|N_{1,1}| = p$. This implies that $|N_{d,1}| = q$, and as always $|N_{d,0}| = 1$.

Plugging equation 5 into equation 3 yields:

$$P_I(t) = \frac{1 - t(\frac{1-q^{-1}-q^{-2}t+q^{-2}t^2+q^{-3}t-q^{-3}t^2}{1-q^{-1}t})}{1-t} \tag{6}$$

We want to manipulate equation 6 so that it becomes $P_I(t) = \sum_{e=0}^{\infty} |N_{d,e}| q^{-2e} t^e$.

Using q^2t as the argument for the Igusa Poincaré Series is advantageous because the q^{-2e} term is canceled out in the summation. Equating both equations for $P_I(q^2t)$:

$$P_I(q^2t) = \sum_{e=0}^{\infty} |N_{d,e}| t^e = \frac{1 - qt^2 + q^2t^2}{1 - qt}$$

Remembering our assumptions about $|N_{d,0}|$ and $|N_{d,1}|$:

$$\begin{aligned} 1 + qT + \sum_{e=2}^{\infty} |N_{d,e}| T^e &= \frac{1 - qT^2 + q^2T^2}{1 - qT} \\ \sum_{e=2}^{\infty} |N_{d,e}| T^e &= \frac{1 - qT^2 + q^2T^2}{1 - qT} - (1 + qT) \end{aligned}$$

Clearing the denominator results in

$$|N_{d,2}| T^2 + \sum_{e=3}^{\infty} (|N_{d,e}| - q|N_{d,e-1}|) T^e = (2q^2 - q) T^2.$$

Thus, from the rational relationship between $P_I(T)$ and $Z_I(T)$ we have the following Lemma:

Lemma 3.1. *Given an elliptic curve with reduction type III over an unramified p-adic extension of degree d:*
$|N_{d,0}| = 1$
$|N_{d,1}| = q$
$|N_{d,2}| = 2q^2 - q$
$|N_{d,e}| = q|N_{d,e-1}|$ *for* $e \geq 3$

¿From this lemma we have the following theorem:

Theorem 3.2. $|N_{d,e}| = 2q^e - q^{e-1}$ *for* $e \geq 2$ *given that* $|N_{d,2}| = 2q^2 - q$

Proof. We will proceed by induction. Our base case for the induction is $e = 2$. From Lemma 1 we have that $|N_{d,2}| = 2q^2 - q$ and this agrees with the formula for $|N_{d,e}|$ when $e = 2$.

Assume $|N_{d,e-1}| = 2q^{e-1} - q^{e-2}$. By Lemma 1 we know that $|N_{d,e}| = q|N_{d,e-1}|$. Using our induction hypothesis and Lemma 1:

$$\begin{aligned} |N_{d,e}| &= q(2q^{e-1} - q^{e-2}) \\ |N_{d,e}| &= 2q^e - q^{e-1} \end{aligned}$$

□

Remembering that $q = p^d$, we can use $|N_{d,e}|$ to find the Weil Poincaré Series, the combined Igusa-Weil Poincaré Series and the Weil Zeta Function for an elliptic curve with Kodaira-Néron Reduction reduction type III quite easily. This is the standard technique for computing the $|N_{d,e}|$ for all reduction types.

4. Case I_0^*

In the * cases we arrive at a polynomial that is a cubic in x when the coefficients are reduced mod p. In I_0^*, we assume that this cubic has three distinct roots in $\overline{\mathbb{F}_p}$. The Igusa zeta function in the dth unramified extension has a term which depends on the number of roots of the cubic in $\mathbb{F}_{p^d}$. This causes a variance in the Igusa zeta function based on $d \bmod 6$ that makes the rest of the calculations more complicated. As this is the most complex case of a phenomenon that occurs in several of the Kodaira-Néron reduction types, it is included to illustrate how we dealt with this issue.

4.1. **The General Igusa Zeta Function.** The authors of [1] computed the Igusa zeta function over $\mathbb{Z}_p$ and their methods completely generalize to a degree d ring of integers. In particular, if we compute the Igusa Zeta Function in a vector space of dimension d over $\mathbb{Z}_p$, and let $q = p^d$, we have

$$Z_I(t) = (1 - q^{-1}) + (1 - q^{-1})q^{-1}t\left(\frac{1 - q^{-1}}{1 - q^{-1}t}\right) + q^{-2}t^2(1 - q^{-1})$$
$$+q^{-3}t^3\left((1 - M_d q^{-1}) + M_d q^{-1}t\left(\frac{1 - q^{-1}}{1 - q^{-1}t}\right)\right),$$

where M_d now depends on the number of roots of the cubic polynomial $f(x) = -x^3 - a_{2,1}x^2 - a_{4,2}x - a_{6,3}$ in $\mathbb{F}_{p^d}$. This difficulty, where we can not simply replace p by p^d when computing the general Igusa zeta function, arises whenever $|N_{d,1}|$ varies within a reduction type based on d. As this occurs in many of the reduction types, the techniques for computing these

4.1.1. *Computing M_d.* M_d is the number of zeroes of the polynomial $f(x) = x^3 - a_{2,1}x^2 - a_{4,2}x - a_{6,3}$ in $\mathbb{F}_p$. In the case I_0^*, we are assuming that $f(x)$ has three distinct roots in $\overline{\mathbb{F}_p}$. For computing M_d, we need to know about the number of roots in $\mathbb{F}_p$ and the number of roots in $\mathbb{F}_{p^d}$.

Theorem 4.1. *If $f(x)$ is an irreducible polynomial of degree d over $\mathbb{F}_p$, then the degree of the splitting field of f is also d.*

For a proof of this theorem, see [3]. This means there are only three possibilities for the splitting field of $f(x)$.

1. $f(x)$ is irreducible over $\mathbb{F}_p$, with splitting field of degree 3.
2. $f(x)$ has exactly one root in $\mathbb{F}_p$.
3. $f(x)$ factors into three linear terms over $\mathbb{F}_p$.

The behavior of $f(x)$ in $\mathbb{F}_p$ completetly characterizes M_d in $\mathbb{F}_q$, since every extension of $\mathbb{F}_p$ of degree d is isomorphic. In the 4 cases listed above, we have

1. If $3|d$, then $M_d = 3$, and if $3 \nmid d$, then $M_d = 0$.
2. If $2|d$, then $M_d = 3$. If $2 \nmid d$, then $M_d = 1$.
3. Regardless of d, $M_d = 3$.

4.2. $|N_{d,e}|$**.** By the usual technique, we find the $|N_{d,e}|$'s.

- $|N_{d,0}| = 1$
- $|N_{d,1}| = q$.
- $|N_{d,2}| = 2q^2 - q$.
- $|N_{d,3}| = 2q^3 - q^2$.
- For $e \geq 4$, $|N_{d,e}| = q^e + M_d q^e - q^{e-1}$.

4.3. **The Generalized Weil Zeta Function.** We have

$$e = 1 \qquad Z_W(W) = \frac{1}{1 - pW}$$

$$e = 2 \qquad Z_W(W) = \frac{1 - pW}{1 - p^2W}$$

$$e = 3 \qquad Z_W(W) = \frac{1 - p^2W}{(1 - p^3W)^2}$$

For $e \geq 4$, the situation is complicated by the presence of the M_d term. We begin by splitting the sum into

$$\exp\left(\sum_{d=1}^{\infty} \frac{((p^e)^d - (p^{e-1})^d)W^d}{d}\right) \exp\left(\sum_{d=1}^{\infty} \frac{M_d (p^e)^d W^d}{d}\right).$$

The first exp can be rewritten in the usual way as $\frac{1-p^{e-1}W}{1-p^eW}$. As noted earlier, M_d depends on the degree of the splitting field of the polynomial $f(x)$.

4.3.1. *$f(x)$ Has Splitting Field of Degree 3.* M_d is now 0 unless $3|d$, in which case $M_d = 3$. The special sum involving M_d thus becomes

$$\exp\left(\sum_{k=1}^{\infty} \frac{3(p^e)^{3k}W^{3k}}{3k}\right) = \frac{1}{1-p^{3e}W^3}$$

$$Z_W(W) = \frac{(1-p^{e-1}W)}{(1-p^eW)(1-p^{3e}W^3)}$$

4.3.2. *$f(x)$ Has Splitting Field of Degree 2.* We have $M_d = 3$ when $d|2$ and $M_d = 1$ else. As in **??**, we can split the exponential into two separate sums and get

$$\exp\left(\sum_{d=1}^{\infty} \frac{(p^e)^d W^d}{d} + \sum_{k=1}^{\infty} \frac{2(p^e)^{2k}W^{2k}}{2k}\right) = \left(\frac{1}{1-p^eW}\right)\left(\frac{1}{1-p^{2e}W^2}\right)$$

$$Z_W(W) = \frac{1-p^{e-1}W}{(1-p^eW)^2(1-p^{2e}W^2)}.$$

4.3.3. *$f(x)$ Factors Completely.* Now $M_d = 3$ regardless of d, giving us

$$\exp\left(\sum_{d=0}^{\infty} \frac{3(p^e)^d W^d}{d}\right) = \left(\frac{1}{1-p^eW}\right)^3$$

$$Z_W(W) = \frac{1-p^{e-1}W}{(1-p^eW)^4}.$$

Computing the other functions we are interested in is straightforward and the results are summarized by a table at the end of this paper.

5. THEOREM ON THE RATIONALITY OF $P(W,T)$

After looking at the various other reduction types of elliptic curves, we see that $P(W,T)$ is always a rational function of W and T. We decide to look at other algebraic curves over $\mathbb{Q}_p$—particularly cubics with singular points. For example, consider the curve $f(x,y) = y^2 - x^3$. This polynomial is singular at $(0,0)$ because the polynomial and both its partial derivatives vanish at this point. Using Igusa's stationary phase formula [2], we compute the Igusa local zeta function associated to this polynomial and find

$$\mathcal{Z}_{\text{Igusa},d}(s) = \frac{(1-p^{-d})(1-p^{-2d}T+p^{-2d}T^2+p^{-5d}T^5)}{(1-p^{-d}T)(1-p^{-5d}T^6)}$$

where as usual $T = p^{-ds}$. Using a relation between the Igusa Poincaré series and our function $P(W,T)$ that will be derived in the proof of the following theorem, we find that

$$P(W,T) = \left(\frac{1}{1-T}\right) \sum_{d=1}^{\infty} \left(\frac{\left(1 + (1-p^{-d})T^2 - T^6\right) W^d}{1 - p^d T^6} \right).$$

This is not a rational function; it has infinitely many poles. For a polynomial in n variables, consider the double Poincaré series

$$P(W,T) = \sum_{d=1}^{\infty} \sum_{e=0}^{\infty} |N_{d,e}| p^{-de(n-1)} T^e W^d.$$

Theorem: Let $f(x)$ be a polynomial in n variables with coefficients in $\mathbb{Z}_p$. Suppose $f(x)$ is nonsingular over $\mathbb{Q}_p$. Then $P(W,T) \in \mathbb{Q}(W,T)$.
Remark: Note that this theorem is much more general than the specific case of elliptic curves that we were previously addressing.

Lemma: The d^{th} Igusa Poincaré series $P_d(T)$ is a rational function of T. Specifically the denominator of $P_d(T)$ is $1 - p^{-d}T$.
Proof of Lemma: Because $f(x)$ is nonsingular over $\mathbb{Q}_p$, given some a in the algebraic variety $\{x \in \mathbb{Q}_p \mid f(x) = 0\}$, we know that there exists some x_i that satisfies

$$\left.\frac{\partial f}{\partial x_i}\right|_{x=a} \neq 0.$$

Because $\partial f/\partial x_i|_{x=a}$ is not zero in $\mathbb{Q}_p$, it is not the case that $\partial f/\partial x_i|_{x=a} \equiv 0 \mod p^e$ for all positive integers e. Therefore, for all e greater than some sufficiently large M, we know that $\partial f/\partial x_i|_{x=a} \not\equiv 0 \mod p^e$.

Now if we think of $f(x)$ over an unramified $K_d/\mathbb{Q}_p$ of degree d, it is still true that $\partial f/\partial x_i|_{x=a} \not\equiv 0 \mod p^e$. Using the multivariable Hensel's lemma over K_d, the cardinality of $N_{d,e}$ satisifies

$$|N_{d,e}| = p^{d(n-1)(e-M)} |N_{d,M}| \qquad \text{for all } e > M.$$

Calculating the d^{th} level Igusa type Poincaré series we see that

$$\begin{aligned} P_d(T) &= \sum_{e=0}^{M-1} |N_{d,e}| p^{-nde} T^e + \sum_{e=M}^{\infty} |N_{d,e}| p^{-nde} T^e \\ &= g(T) + |N_{d,M}| \sum_{e=M}^{\infty} p^{nde-de-ndM+dM} p^{-nde} T^e \\ &= \frac{g(T)(1-p^{-d}T) + |N_{d,M}| p^{ndM} T^M}{1-p^{-d}T} \end{aligned}$$

where $g(T)$ is just another name for the polynomial $\sum_{e=0}^{M-1}|N_{d,e}|p^{-nde}T^e$. We see that the numerator of $P_d(T)$ is just some polynomial in T with coefficients in $\mathbb{Q}$ and the denominator is $1-p^{-d}T$. This proves our lemma.

Proof of Theorem: First we manipulate our double Poincaré series. We see that

$$\begin{aligned}P(W,T) &= \sum_{d=1}^{\infty}\sum_{e=0}^{\infty}|N_{d,e}|p^{-de(n-1)}T^eW^d = \sum_{d=1}^{\infty}W^d\sum_{e=0}^{\infty}|N_{d,e}|p^{-nde}p^{de}T^e \\ &= \sum_{d=1}^{\infty}W^d\sum_{e=0}^{\infty}|N_{d,e}|p^{-nde}(p^dT)^e = \sum_{d=1}^{\infty}W^dP_d(p^dT).\end{aligned}$$

By our lemma, we know the exact form of $P_d(T)$. We can rewrite our double Poincaré series in terms of $P_d(p^dT)$ to get

$$\begin{aligned}P(W,T) &= \left(\frac{1}{1-T}\right)\sum_{d=1}^{\infty}\left(W^dg(p^dT)(1-T)+|N_{d,M}|p^{ndM+dM}W^dT^M\right) \\ &= \sum_{d=1}^{\infty}\sum_{e=0}^{M-1}|N_{d,e}|p^{-nde+de}T^eW^d+\left(\frac{T^M}{1-T}\right)\sum_{d=1}^{\infty}|N_{d,M}|(p^{nM+M}W)^d \\ &= \sum_{e=0}^{M-1}T^eP_{\mathrm{Weil},\,e}(p^{-ne+e}W)+\left(\frac{T^M}{1-T}\right)P_{\mathrm{Weil},\,e}(p^{nM+M}W).\end{aligned}$$

Diane Meuser showed in [4] that all the analygous Weil zeta functions defined by

$$\mathcal{Z}_{\mathrm{Weil},\,e}(W) = \exp\sum_{d=1}^{\infty}|N_{d,e}|\frac{W^d}{d}$$

are always rational functions of W. This in turn implies the rationality of any Weil type Poincaré series $P_{\mathrm{Weil},\,e}(W)$. So in our above representation of $P(W,T)$, we have a finite sum of rational functions in W and T. Thus $P(W,T)\in\mathbb{Q}(W,T)$. □

We now investigate the converse of this theorem—unfortunately it is not true in general. Consider the singular curve $f(x,y)=xy$. After counting carefully, we find that

$$|N_{d,e}| = (e+1)p^{ed}-ep^{d(e-1)}.$$

Using this information, we find

$$P(W,T) = \frac{W(TW-T-W+p)}{(1-T)^2(1-W)(p-W)}$$

This is indeed a rational function of W and T although our curve $f(x,y)$ is singular at $(0,0)$.

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Table 1. $|N_{d,e}|$ for Elliptic Curves by Reduction Type

Reduction Type	$\|N_{d,e}\|$
I_0	$\|N_{d,e}\| = p^{de} - p^{d(e-1)}(\beta_1^d + \beta_2^d) = \|N_{d,1}\|pp^{d(e-1)}$
I_n	for n even, $1 \leq e \leq 2+2m : \|N_{d,e}\| = ep^{de} - ep^{d(e-1)}$ for $e \geq 3+2m : \|N_{d,e}\| = (2+2m)p^{de} - (3+2m)p^{d(e-1)}$ for n odd, $\|N_{d,1}\| = p^d - 1$ $\|N_{d,2}\| = 2p^{2d} - 2p^d$ for $1 \leq e < 2m+2 : \|N_{d,e}\| = ep^{de} - ep^{d(e-1)}$ $\|N_{d,2m+2}\| = (2m+1)p^{d(2m+2)} - (2m+2)p^{d(2m+1)}$ $\|N_{d,2m+3}\| = (2m+1)p^{d(2m+3)} - (2m+2)p^{d(2m+2)}$ for $e \geq 2m+4$ $\|N_{d,e}\| = (2m+1)p^{de} - (2m+2)p^{d(e-1)}$
II	$\|N_{d,1}\| = p^d$ for $e \geq 2 : \|N_{d,e}\| = p^{de} - p^{d(e-1)}$
III	$\|N_{d,1}\| = p^d$ for $e \geq 2 : \|N_{d,e}\| = 2p^{de} - p^{d(e-1)}$
IV	$\|N_{d,1}\| = p^d$ $\|N_{d,2}\| = p^d(2p^d - 1)$ for $e \geq 3$ Subcase A: $p^{d(e-1)}(3p^d - 1)$ Subcase B: $p^{d(e-1)}(p^d - 1)$
I_0*	$\|N_{d,1}\| = p^d$ $\|N_{d,2}\| = 2p^{2d} - p^d$ $\|N_{d,3}\| = 2p^{3d} - p^{2d}$ $\|N_{d,4}\| = p^{4d}M - p^{4d}$ for $e \geq 5 : \|N_{d,e}\| = p^{de} + Mp^{de} - p^{d(e-1)}$
I_n*	$\|N_{d,1}\| = p^d$ $\|N_{d,2}\| = 2p^{2d} - p^d$ $\|N_{d,3}\| = 2p^{3d} - p^{2d}$ $\|N_{d,4}\| = 3p^{4d} - p^{3d}$ for $4 \leq e \leq n+3 : \|N_{d,e}\| = 3p^{de} - p^{d(e-1)}$ for $e \geq n+4$ $\|N_{d,e}\| = 4p^{de} - p^{d(e-1)}$
$II*$	$\|N_{d,1}\| = p^d$ $\|N_{d,2}\| = 2p^{2d} - p^d$ $\|N_{d,3}\| = 2p^{3d} - p^{2d}$ $\|N_{d,4}\| = 2p^{4d} - p^{3d}$ $\|N_{d,5}\| = 2p^{5d} - p^{4d}$ $\|N_{d,6}\| = p^{6d} - p^{5d}$ for $e \geq 7$ $\|N_{d,e}\| = p^{de} - p^{d(e-1)}$
$III*$	same as Type III
$IV*$	with a triple root, $\|N_{d,1}\| = p^d$ for $2 \leq e \leq 4 : \|N_{d,e}\| = p^{d(e-1)}(2p^d - 1)$ for $e \geq 5 : \|N_{d,e}\| = p^{d(e-1)}(3p^d - 1)$ with distinct roots, $\|N_{d,1}\| = p^d$ for $2 \leq e \leq 4 : \|N_{d,e}\| = p^{d(e-1)}(2p^d - 1)$ for $e \geq 5$ $\|N_{d,e}\| = p^{d(e-1)}(p^d - 1)$

Table 2. Super Poincaré Series for Elliptic Curves by Reduction Type

Reduction Type	Super Poincaré Series
I_0	$P(T,W) = \frac{W}{1-W} + \frac{T}{1-T}(P_{Weil}(p^{-1}W)$
I_n	*extended results*
II	$P(T,W) = \frac{W-p^{-1}W^2-p^{-1}T^2W+p^{-1}T^2W^2}{(1-T)(1-W)(1-p^{-1}W)}$
III	$P(T,W) = \frac{W+T^2W-p^{-1}T^2W-p^{-1}W^3}{(1-T)(1-W)(1-p^{-1}W)}$
IV	Subcase A: $P(T,W) = \frac{W(-T^2+p(1+T^2+T^3)-W-T^3W)}{(T-1)(p-W)(W-1)}$ Subcase B: $P(T,W) = \frac{W(-T^2+p(1+T^2-T^3)-W+T^3W)}{(T-1)(p-W)(W-1)}$
I_0*	*extended results*
I_n*	Case 1: $P(T,W) = \frac{(W-W^2p^{-1})T^{n+4}+(W-Wp^{-1})T^4+(Wp^{-1}-2p^{-1}W^2+W)T^2}{(1-T)(1-W)(1-p^{-1}W)}$ $+\frac{(W-p^{-1}W)T}{(1-T)(1-W)(1-p^{-1}W)}$ Case 2: $P(T,W) = \frac{(-W^3p^{-1}+W^2p^{-1}+W^2-W)T^{n+4}+(W^2-p^{-1}W^2+W-p^{-1}W)T^4}{(1-T)(1-W)(1+W)(1-p^{-1}W)}$
$II*$	$P(T,W) = \frac{W-p^{-1}W)T^2(1+T^2)(1+T)}{(1-W)(1-p^{-1}W)}$
$III*$	same as Type III
$IV*$	Case 1: $P(W,T) = \frac{TW+W+2T^2W+2T^3W+2T^4W}{1-W} - \frac{pWT^2(1+T+T^2)}{p-W} + \frac{T^5 3W+pT^5(W-W^2-3)}{(T-1)(W-1)(W-p)}$ Case 2: $P(T,W) = \frac{W+TW+2T^2W2T^3W+2T^4W}{1-W} - \frac{T^2W+T^3W+T^4W}{1-p^{-1}W} + \frac{3T^5W+T^5p(W-3-W^2)}{(T-1)(W-1)(W-p)}$