

Computing the $|N_{d,e}|$'s for a Reduction Type II Elliptic Curve

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The 1999 REU Group found the Igusa local zeta function for an elliptic curve of reduction type II to have the rational form:

$$Z(T) = (p^2 - p)p^{-2} + (p - 1)p^{-2}T \frac{1 - p^{-1}}{1 - p^{-1}T} + p^{-2}T \quad (1)$$

We also know that the Poincaré Series can be expressed in two different forms:

$$P(T) = \frac{1 - TZ(T)}{1 - T} \quad (2)$$

$$P(T) = \sum_{e=0}^{\infty} |N_{d,e}| q^{-2e} T^e \quad (3)$$

Let's now assume that we can change all of the p 's in equation (1) to p^d 's when we are working in an unramified extension field, K , of degree d over \mathbb{Q}_p , and let's let $p^d = q$. Thus in K , our new Igusa Local Zeta Function becomes:

$$Z(T) = (q^2 - q)q^{-2} + (q - 1)q^{-2}T \frac{1 - q^{-1}}{1 - q^{-1}T} + q^{-2}T \quad (4)$$

Now all we have to do to solve for the $|N_{d,e}|$ is plug equation (4) into equation (2) and expand it into an infinite sum. Once we have this we can relate it to equation (3) and solve for the $|N_{d,e}|$. The 1999 REU Group found $|N_{1,1}| = p$. So we will assume that $|N_{d,1}| = q$, and as always $|N_{d,0}| = 1$.

Plugging equation (4) into equation (2) we get:

$$P(T) = \frac{1 - T[(q^2 - q)q^{-2} + (q - 1)q^{-2}T \frac{1 - q^{-1}}{1 - q^{-1}T} + q^{-2}T]}{1 - T} \quad (5)$$

We want to manipulate equation (5) so that it becomes $P(T) = \sum_{e=0}^{\infty} |N_{d,e}| q^{-2e} T^e$. If we plug in $q^2 T$ for T in equation (3) we see that:

$$P(q^2 T) = \sum_{e=0}^{\infty} |N_{d,e}| q^{-2e} (q^2 T)^e$$

$$\begin{aligned}
&= \sum_{e=0}^{\infty} |N_{d,e}| q^{-2e} q^{2e} T^e \\
&= \sum_{e=0}^{\infty} |N_{d,e}| T^e
\end{aligned}$$

So plugging in q^2T in for T in the Poincaré Series is advantageous because we don't have to worry about the q^{-2e} term in the summation. Finding $P(q^2T)$ for equation (5):

$$\begin{aligned}
P(q^2T) &= \frac{1 - q^2T[(q^2 - q)q^{-2} + (q - 1)q^{-2}q^2T \frac{1-q^{-1}}{1-q^{-1}q^2T} + q^{-2}q^2T]}{1 - q^2T} \\
&= \frac{1 - q^2T[1 - q^{-1} + (q - 1)T \frac{1-q^{-1}}{1-qT} + T]}{1 - q^2T} \\
&= \frac{1 - q^2T + qT - q^2T^2(q - 1) \frac{1-q^{-1}}{1-qT} - q^2T^2]}{1 - q^2T} \\
&= \frac{1 - q^2T + qT - q^2T^2}{1 - q^2T} - \frac{q^2T^2(q - 2 + q^{-1})}{(1 - qT)(1 - q^2T)} \\
&= \frac{1 - q^2T + qT - q^2T^2}{1 - q^2T} + \frac{-q^3T^2 + 2q^2T^2 - qT^2}{(1 - qT)(1 - q^2T)}
\end{aligned}$$

Thus equating both equations for $P(q^2T)$:

$$\sum_{e=0}^{\infty} |N_{d,e}| T^e = \frac{1 - q^2T + qT - q^2T^2}{1 - q^2T} + \frac{-q^3T^2 + 2q^2T^2 - qT^2}{(1 - qT)(1 - q^2T)}$$

Remembering our assumptions about $|N_{d,0}|$ and $|N_{d,1}|$:

$$\begin{aligned}
|N_{d,0}|T^0 + |N_{d,1}|T^1 + \sum_{e=2}^{\infty} |N_{d,e}|T^e &= \frac{1 - q^2T + qT - q^2T^2}{1 - q^2T} + \frac{-q^3T^2 + 2q^2T^2 - qT^2}{(1 - qT)(1 - q^2T)} \\
1 + qT + \sum_{e=2}^{\infty} |N_{d,e}|T^e &= \frac{1 - q^2T + qT - q^2T^2}{1 - q^2T} + \frac{-q^3T^2 + 2q^2T^2 - qT^2}{(1 - qT)(1 - q^2T)} \\
\sum_{e=2}^{\infty} |N_{d,e}|T^e &= \frac{1 - q^2T + qT - q^2T^2}{1 - q^2T} + \frac{-q^3T^2 + 2q^2T^2 - qT^2}{(1 - qT)(1 - q^2T)} - (1 + qT)
\end{aligned}$$

Now let's multiply through by $1 - q^2T$:

$$\begin{aligned}
(1 - q^2T) \sum_{e=2}^{\infty} |N_{d,e}|T^e &= 1 - q^2T + qT - q^2T^2 + \frac{-q^3T^2 + 2q^2T^2 - qT^2}{1 - qT} - (1 + qT)(1 - q^2T) \\
(1 - q^2T) \sum_{e=2}^{\infty} |N_{d,e}|T^e &= 1 - q^2T + qT - q^2T^2 + \frac{-q^3T^2 + 2q^2T^2 - qT^2}{1 - qT} - 1 + q^2T - qt + q^3T^2
\end{aligned}$$

$$(1 - q^2T) \sum_{e=2}^{\infty} |N_{d,e}| T^e = q^3T^2 - q^2T^2 + (-q^3T^2 + 2q^2T^2 - qT^2) \frac{1}{1 - qT}$$

Remember that the fraction $\frac{1}{1 - qT} = \sum_{e=0}^{\infty} (qT)^e$ so:

$$(1 - q^2T) \sum_{e=2}^{\infty} |N_{d,e}| T^e = q^3T^2 - q^2T^2 + (-q^3T^2 + 2q^2T^2 - qT^2) \sum_{e=0}^{\infty} (qT)^e \quad (6)$$

Now let's take a look at the left-hand side of the equation:

$$\begin{aligned} (1 - q^2T) \sum_{e=2}^{\infty} |N_{d,e}| T^e &= \sum_{e=2}^{\infty} |N_{d,e}| T^e - q^2T \sum_{e=2}^{\infty} |N_{d,e}| T^e \\ &= \sum_{e=2}^{\infty} |N_{d,e}| T^e - q^2 \sum_{e=2}^{\infty} |N_{d,e}| T^{e+1} \\ &= |N_{d,2}| T^2 + \sum_{e=3}^{\infty} |N_{d,e}| T^e - q^2 \sum_{e=3}^{\infty} |N_{d,e-1}| T^e \\ &= |N_{d,2}| T^2 + \sum_{e=3}^{\infty} (|N_{d,e}| - q^2 |N_{d,e-1}|) T^e \end{aligned}$$

Now referring back to equation (6):

$$\begin{aligned} |N_{d,2}| T^2 + \sum_{e=3}^{\infty} (|N_{d,e}| - q^2 |N_{d,e-1}|) T^e &= q^3T^2 - q^2T^2 + (-q^3T^2 + 2q^2T^2 - qT^2) \sum_{e=0}^{\infty} (qT)^e \\ &= q^3T^2 - q^2T^2 + (-q^3T^2 + 2q^2T^2 - qT^2) \sum_{e=2}^{\infty} (qT)^{e-2} \\ &= q^3T^2 - q^2T^2 + \sum_{e=2}^{\infty} (-q^3T^2 + 2q^2T^2 - qT^2) (qT)^{e-2} \\ &= q^3T^2 - q^2T^2 + \sum_{e=2}^{\infty} (-q^3 + 2q^2 - q) q^{e-2} T^e \\ &= q^3T^2 - q^2T^2 + \sum_{e=2}^{\infty} (-q^{e+1} + 2q^e - q^{e-1}) T^e \end{aligned}$$

If we pull out the $e = 2$ term from the right-hand sum we get:

$$\begin{aligned} |N_{d,2}| T^2 + \sum_{e=3}^{\infty} (|N_{d,e}| - q^2 |N_{d,e-1}|) T^e &= q^3T^2 - q^2T^2 + T^2(-q^3 + 2q^2 - q) + \sum_{e=3}^{\infty} (-q^{e+1} + 2q^e - q^{e-1}) T^e \\ |N_{d,2}| T^2 + \sum_{e=3}^{\infty} (|N_{d,e}| - q^2 |N_{d,e-1}|) T^e &= (q^2 - q) T^2 + \sum_{e=3}^{\infty} (-q^{e+1} + 2q^e - q^{e-1}) T^e \end{aligned}$$

Finally we have $|N_{d,2}| = q^2 - q$ and $|N_{d,e}| - q^2|N_{d,e-1}| = -q^{e+1} + 2q^e - q^{e-1}$. From what we know about $|N_{d,2}|$, let's assume that $|N_{d,e}| = q^e - q^{e-1}$. To prove this by induction, we have to prove that $|N_{d,e-1}| = q^{e-1} - q^{e-2}$.

Let's plug what we assume to be $|N_{d,e}|$ into our equation $|N_{d,e}| - q^2|N_{d,e-1}| = -q^{e+1} + 2q^e - q^{e-1}$. We get:

$$\begin{aligned} q^e - q^{e-1} - q^2|N_{d,e-1}| &= -q^{e+1} + 2q^e - q^{e-1} \\ -q^2|N_{d,e-1}| &= -q^{e+1} + q^e \\ |N_{d,e-1}| &= q^{e-1} - q^{e-2} \end{aligned}$$

This is exactly what we wanted. So we have proved that for a Reduction Type II Elliptic Curve the $|N_{d,e}| = q^e - q^{e-1}$, where $q = p^d$.