Rationality of the Weil-Igusa type Poincaré Series

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Let $f(x_1, ..., x_n)$ be a polynomial in n variables with coefficients in $\mathbb{Z}_p$ for a fixed prime p. Consider the set

$$N_{d,e} = \left\{ x \in \mathcal{O}_d / p^e \mathcal{O}_d^{(n)} \,\middle|\, f(x) \equiv 0 \mod p^e \right\}.$$

We are interested in the cardinality of $N_{d,e}$, denoted $|N_{d,e}|$. To study these cardinalities, we consider the double Poincaré series in two variables given by

$$P(W, T) = \sum_{d=1}^{\infty} \sum_{e=0}^{\infty} |N_{d,e}| p^{-de(n-1)} W^d T^e.$$

This Poincaré series is very similar to one that Diane Meuser worked with in her paper *The meromorphic continuation of a zeta function of Weil and Igusa type*. This is also a generalization of the double Poincaré series that we have been working with during this REU. When $f(x)$ is an algebraic curve, that is when $f(x)$ is a polynomial in $n = 2$ variables, then $P(W, T)$ agrees with the series we have studied.

As we shall see, $P(W, T)$ is closely related to the d^{th} level Igusa type Poincaré series defined by

$$P_d(T) = \sum_{e=0}^{\infty} |N_{d,e}| p^{-nde} T^e.$$

In general $P(W, T)$ is not a rational function of W and T. However, the following theorem is nice.

Theorem: Suppose $f(x)$ is nonsingular over $\mathbb{Q}_p$. Then $P(W, T) \in \mathbb{Q}(W, T)$.

Lemma: $P_d(T)$ is a rational function. Specifically the denominator of $P_d(T)$ is $1 - p^{-d}T$.

Proof of Lemma: Because $f(x)$ is nonsingular over $\mathbb{Q}_p$, given some a in the algebraic variety $\{x \in \mathbb{Q}_p \,|\, f(x) = 0\}$, we know that there exists some x_i such that

$$\left.\frac{\partial f}{\partial x_i}\right|_{x=a} \neq 0.$$

Because $\partial f/\partial x_i|_{x=a}$ is not zero in $\mathbb{Q}_p$, it is not the case that $\partial f/\partial x_i|_{x=a} \equiv 0 \mod p^e$ for all positive integers e. Therefore, for all e greater than some sufficiently large M, we know that $\partial f/\partial x_i|_{x=a} \not\equiv 0 \mod p^e$.

Now if we think of $f(x)$ over an unramified $K_d/\mathbb{Q}_p$ of degree d, it is still true that $\partial f/\partial x_i|_{x=a} \not\equiv 0 \mod p^e$. Using the multivariable Hensel's lemma over K_d, the cardinality of $N_{d,e}$ satisify

$$|N_{d,e}| = p^{d(n-1)(e-M)}|N_{d,M}| \qquad \text{for all } e > M.$$

Calculating the d^{th} level Igusa type Poincaré series we see that

$$\begin{aligned} P_d(T) &= \sum_{e=0}^{M-1} |N_{d,e}|p^{-nde}T^e + \sum_{e=M}^{\infty} |N_{d,e}|p^{-nde}T^e \\ &= g(T) + |N_{d,M}| \sum_{e=M}^{\infty} p^{nde-de-ndM+dM}p^{-nde}T^e \\ &= g(T) + |N_{d,M}|p^{ndM+dM} \sum_{e=M}^{\infty} (p^{-d}T)^e \\ &= g(T) + |N_{d,M}|p^{ndM+dM} \frac{(p^{-d}T)^M}{1-p^{-d}T} \\ &= \frac{g(T)(1-p^{-d}T) + |N_{d,M}|p^{ndM}T^M}{1-p^{-d}T} \end{aligned}$$

where $g(T)$ is just another name for the polynomial $\sum_{e=0}^{M-1} |N_{d,e}|p^{-nde}T^e$. We see that the numerator of $P_d(T)$ is just some polynomial in T with coefficients in $\mathbb{Q}$ and the denominator is $1 - p^{-d}T$. This completes our proof.

Proof of Theorem: First we play around with our double Poincaré series. We see that

$$P(W,T) = \sum_{d=1}^{\infty}\sum_{e=0}^{\infty} |N_{d,e}|p^{-de(n-1)}T^eW^d = \sum_{d=1}^{\infty} W^d \sum_{e=0}^{\infty} |N_{d,e}|p^{-nde}p^{de}T^e$$

$$= \sum_{d=1}^{\infty} W^d \sum_{e=0}^{\infty} |N_{d,e}|p^{-nde}(p^dT)^e = \sum_{d=1}^{\infty} W^d P_d(p^dT).$$

By our lemma, we know the exact form of $P_d(T)$. We can rewrite our double

Poincaré series in terms of $P_d(p^d T)$ to get

$$\begin{aligned}
P(W,T) &= \left(\frac{1}{1-T}\right)\sum_{d=1}^{\infty}\left(W^d g(p^d T)(1-T) + |N_{d,M}|p^{ndM+dM}W^d T^M\right) \\
&= \sum_{d=1}^{\infty} W^d g(p^d T) + \left(\frac{T^M}{1-T}\right)\sum_{d=1}^{\infty}|N_{d,M}|p^{ndM+dM}W^d \\
&= \sum_{d=1}^{\infty}\sum_{e=0}^{M-1}|N_{d,e}|p^{-nde+de}T^e W^d + \left(\frac{T^M}{1-T}\right)\sum_{d=1}^{\infty}|N_{d,M}|(p^{nM+M}W)^d \\
&= \sum_{e=0}^{M-1} T^e \sum_{d=1}^{\infty}|N_{d,e}|(p^{-ne+e}W)^d + \left(\frac{T^M}{1-T}\right)\sum_{d=1}^{\infty}|N_{d,M}|(p^{nM+M}W)^d \\
&= \sum_{e=0}^{M-1} T^e P_{\text{Weil},\,e}(p^{-ne+e}W) + \left(\frac{T^M}{1-T}\right)P_{\text{Weil},\,e}(p^{nM+M}W).
\end{aligned}$$

Diane Meuser showed that all the analygous Weil zeta functions defined by

$$\mathcal{Z}_{\text{Weil},\,e}(W) = \exp\sum_{d=1}^{\infty}|N_{d,e}|\frac{W^d}{d}$$

are always rational functions of W. This in turn implies the rationality of any Weil type Poincaré series $P_{\text{Weil},\,e}(W)$. So in our above representation of $P(W,T)$, we have a finite sum of rational functions in W and T. Thus $P(W,T) \in \mathbb{Q}(W,T)$.