Summer Research

Topic: The Random Path Generated By a Particle Bombarded By Molecules

MHC Math / Stats Department
By Karen & Young
Introduction: What is Brownian Motion?

If you put drops of food coloring in water

- **Macro:** diffusion process -- coloring spread out over time
- **Micro:** atomic collision process -- coloring particle is bouncing around with the water molecules

The motion of independent particles in 2D plane is a Brownian Motion
Brownian Motion

- A continuous-time, continuous-state stochastic process \( (B_t)_{t \geq 0} \)
- \( B_t \) is a random variable for each time \( t \), describes position of diffusion particle
- Two important properties
  - Normal Distribution: for \( t > 0, B_t \sim N(0, t) \)
  - Continuous paths: as r.v moves in space, no sudden jumps happen
Our Model

- A microscopic picture of the macro process
- Studies the random path generated by a distinguished particle after collision with molecules

Constraint on the particle: rolling without slipping / twisting
Simple Model (I) : One Step of the Rolling Process

Unit sphere moves with

- Instantaneous random velocity $v$
- 4 directions $\xi_1, \xi_2, \xi_3, \xi_4$ with equal probability of $P(v = \xi_k) = \frac{1}{4}$
- Fixed time interval $t$

Position of $C_1 = C_0 + t \ v$
Simple Model (II): Multiple Random Steps

After one step is completed, the ball rolls again in a randomly chosen new direction.

- $v_n$: i.i.d sequence with the same distribution as $v$ (4 directions with equal probability)
- Position of $C_n = C_0 + \sum t \cdot v_n$
Positions of C and N

Observe the paths generated by the center $C$ of the sphere and an arbitrary point (e.g. north pole $N$) on sphere

\[
\text{Position of } C_n = C_0 + \sum t v_n
\]

\[
\text{Position of } N_n = C_n + X_n
\]

Rotational velocity is transformed from linear velocity, expressed in matrix $Q$

- $X_n = e^{tQ} X_{n-1}$
- $N_n = C_n + X_n$
Simulation

Paths of the center (black) and arbitrary point (blue)

Eg. steps $n = 1500$

Arbitrary point and center follow similar patterns in their paths
Theorem

- Previous simulation path of the center is an approximation of a Brownian Motion

- $C_t^h =$ position of the center after time $t$

- The distribution of $C_t^h$ converges to the distribution of $\sigma_* B_t$
  - $B_t$ is a 2D Brownian motion with variance $\sigma_*^2$
  - $\sigma_* = \sigma_h / 2$
  - $\sigma_h$: jump size
## Approximation of $\sigma_* B_t$: Distribution Converges

<table>
<thead>
<tr>
<th>$\chi_t^h$</th>
<th>$\sigma_* B_t$</th>
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<tbody>
<tr>
<td>Can be simulated in fine detail</td>
<td>Well understood theoretically</td>
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<tr>
<td>We control micro details of the model</td>
<td>All micro details summarized/obscured in $\sigma_*$</td>
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Simple Model (III) : Understanding micro via macro

- Our experiment: measure the (mean) escape time from a given region $E(T_h)$
- Goal: understand the effects of some microscopic parameters on the escape time

**Escape time**: first time particle escapes region of radius $R$

$$T = \inf \{ t > 0 : |C_t| \geq R \}$$
**Simulation : Escape Time**

**Question :** How does the escape time depend on velocity $\xi$?

**Refinement :** How does mean escape time $E(T_h)$ depend on $\sigma_h$?

1) Keeping range $R$ fixed,
2) Let a range of $\sigma_h$ values,
3) Compute $E(T_h)$ for each value of $\sigma_h$
4) Plot $\sigma_h - E(T_h)$
Simulation: Relationship between $E(T)$ and $\sigma_h$

$$E(T_h) \propto 1 / \sigma_h^2$$
**Simple Model (III): Exploring $X_t^h$ by understanding $\sigma^*_B t$**

<table>
<thead>
<tr>
<th>Simulation</th>
<th>Theory</th>
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<tbody>
<tr>
<td>Discrete Random Walk</td>
<td>Brownian Motion</td>
</tr>
<tr>
<td>Escape Time for Random Walk</td>
<td>Escape Time for B.M (Dynkin’s formula):</td>
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<tr>
<td>$T_h$: Escape time related to $h$</td>
<td>$\text{Escape Time} = R^2 / n\sigma_*^2$</td>
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**Corollary:**

Let $T_h = \inf \{ t > 0 : |X_t^h| \geq R \}$, then $\lim_{h \to 0} E[T_h] = R^2 / n\sigma_*^2$
Ongoing Work: More Generalized Version

- A model with a general /continuous state space of directions
- Changes in direction comes from interaction with random motion of small molecules
- The velocity of the ball carries a memory of previous velocities \((m.c \ vs \ r.w)\)
- Investigate how escape time is affected by mass, air density and temperature
Escape Time
Escape Time
Special Thanks To Professor Tim Chumley