Math 211-01, Spring 2017 — Exam 2

Mount Holyoke College

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Instructions. The exam consists of 5 questions, some with multiple parts, for a total of 50 points. Please do all of them and **please mark your answers clearly**. To receive full credit, you must show your work and provide details and justification where appropriate. You may use an 8.5×11 inch sheet of notes. No other materials or devices are allowed. The last page of the exam is left blank for scratch work.

Name: ______________________________

Problem	Score
1	
2	
3	
4	
5	
Total	

Problem 1 (10 points). Let

$$A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 5 & 0 & 0 \\ 1 & 0 & 3 & -2 \\ 1 & 2 & 0 & 0 \\ 9 & 1 & 0 & 2 \end{bmatrix}.$$

1. Compute $\det A$.

$$\det A = \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} = \boxed{-10}$$

2. Compute $\det B$.

$$\det B = -3 \det \begin{bmatrix} 5 & 5 & 0 \\ 1 & 2 & 0 \\ 9 & 1 & 2 \end{bmatrix} = (-3)(2) \det \begin{bmatrix} 5 & 5 \\ 1 & 2 \end{bmatrix}$$

$$= (-3)(2)(10-5) = \boxed{-30}$$

3. Compute $\det AB$.

$$\det AB = \det A \det B = (-10)(-30) = \boxed{300}$$

4. Compute $\det B^{-1}$.

$$\det B^{-1} = \frac{1}{\det B} = \boxed{-\frac{1}{30}}$$

Problem 2 (8 points). Let

$$A = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & -1 & 1 & -1 \\ 0 & -3 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Find each of the following.

1. A basis for im A.

first, second, fourth columns of A are a basis of im(A) (found by looking for leading 1 columns in rref(A)).

$$\boxed{\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -3 \\ 1 \end{bmatrix} \right\}}$$

2. A basis for ker A.

ker(A) consists of $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ such that

$$\begin{aligned} x_1 &= 0 \\ x_2 - x_3 &= 0 \\ x_4 &= 0 \\ x_3 & \text{ free} \end{aligned} \implies \vec{x} = \begin{bmatrix} 0 \\ t \\ t \\ 0 \end{bmatrix}, \quad t \in \mathbb{R}$$

$\implies \boxed{\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\} \text{ is a basis}}.$

3. det A.

$\boxed{\det A = 0}$ since $\ker(A) \neq \{\vec{0}\}$ implies A is not invertible.

Problem 3 (10 points). Let V be the plane given by the equation

$$x_1 - x_2 + x_3 = 0$$

and let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be orthogonal projection onto V.

1. Find a basis $\{\vec{v}_1, \vec{v}_2\}$ for V.

any two linearly indep vectors which satisfy the above equation

for example, $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

2. Describe $\ker T$ geometrically.

The line through origin spanned by

$\vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

3. Describe $\operatorname{im} T$ geometrically.

The plane V itself.

4. Let $\vec{v}_3$ be vector perpendicular to and let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be the basis of $\mathbb{R}^3$ which consists of the basis vectors for V found in part 1 along with $\vec{v}_3$. Find the $\mathcal{B}$-matrix of T.

$T(\vec{v}_1) = \vec{v}_1$

$T(\vec{v}_2) = \vec{v}_2$ so $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$T(\vec{v}_3) = \vec{0}$

Problem 4 (10 points). Short answer questions.

1. Explain why you cannot find an $n \times n$ matrix A where $\operatorname{im} A = \ker A$ when n is odd.

If $\operatorname{im}(A) = \ker(A)$, then $\dim(\operatorname{im}(A)) + \dim(\ker(A)) = 2\dim(\operatorname{im}(A))$ is even. But by rank-nullity theorem, this sum must be n, which is assumed odd. This is a contradiction.

2. Give an example of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ whose image is the x_1-axis in $\mathbb{R}^2$.

projection onto line spanned by $\vec{e}_1$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}.$$

3. Give an example of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ whose image is all of $\mathbb{R}^3$.

any invertible linear transformation $\mathbb{R}^3 \to \mathbb{R}$

such as reflection across a plane.

4. Suppose that $\vec{u}, \vec{v}, \vec{w}$ are linearly independent vectors. Show that

$$\vec{u} + \vec{w}, 2\vec{v}, \vec{u} - \vec{w}$$

are linearly independent.

Suppose $c_1(\vec{u} + \vec{w}) + c_2(2\vec{v}) + c_3(\vec{u} - \vec{w}) = \vec{0}$

Then $(c_1 + c_3)\vec{u} + (2c_2)\vec{v} + (c_1 - c_3)\vec{w} = \vec{0}$

Since $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly indep.

$$\begin{aligned} c_1 + c_3 &= 0 \\ 2c_2 &= 0 \\ c_1 - c_3 &= 0 \end{aligned} \implies c_2 = 0 \text{ and } \begin{aligned} c_1 &= -c_3 \\ c_1 &= c_3 \end{aligned} \implies c_1 = -c_1 \implies c_1 = 0, \; c_3 = 0$$

Thus $\{\vec{u} + \vec{w}, 2\vec{v}, \vec{u} - \vec{w}\}$ is linearly independent.

Problem 5 (12 points). For each of the following statements, say whether the statement is true or false and give justification or a counterexample.

1. The set of vectors $V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_1 \leq 1, x_2 \leq 1, x_3 \leq 1 \right\}$ forms a subspace of $\mathbb{R}^3$.

$\boxed{F}$ let $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in V$ and $k = 2$

then $k\vec{x} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \notin V$ so V not closed under scalar multiplication

2. The vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \\ -7 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 9 \\ 8 \\ 7 \\ 6 \end{bmatrix}$ are linearly independent.

$\boxed{F}$ any $\underline{5}$ vectors in $\mathbb{R}^4$ must be linearly dependent

3. If the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ span $\mathbb{R}^3$ then n must be equal to 3.

$\boxed{F}$ n could be greater than 3

(but $\vec{v}_1, \ldots, \vec{v}_n$ would be linearly dependent).

4. If A is an invertible square matrix, then $\det A = \det(\text{rref } A)$.

F

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad \det(A) = 27$$

$\text{rref}(A) = I_3$, $\det(\text{rref}(A)) = 1$

5. There exists a 3×3 matrix A such that $\det A = \det(-A)$.

T

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

6. There exists an invertible matrix of the form $\begin{bmatrix} a & e & f & j \\ b & 0 & g & 0 \\ c & 0 & h & 0 \\ d & 0 & k & 0 \end{bmatrix}$.

F

Cannot be invertible since columns 2 and 4 are scalar multiples.

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