Math 211, Spring 2017 — Homework 1

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Due February 2

Problem 1 (Exercises 1.1.11, 12, 13)**.** Find all solutions of the following linear systems. Represent your solutions graphically, as intersections of lines in the xy-plane.

1.

$$\begin{aligned} x - 2y &= 2 \\ 3x + 5y &= 17 \end{aligned}$$

2.

$$\begin{aligned} x - 2y &= 3 \\ 2x - 4y &= 6 \end{aligned}$$

3.

$$\begin{aligned} x - 2y &= 3 \\ 2x - 4y &= 8 \end{aligned}$$

Problem 2 (Exercise 1.1.19)**.** Consider the linear system

$$\begin{aligned} x + y - z &= -2 \\ 3x - 5y + 13z &= 18 \\ x - 2y + 5z &= k, \end{aligned}$$

where k is an arbitrary number.

1. For which value(s) of k does this system have one or infinitely many solutions?
2. For each value of k you found in part a, how many solutions does the system have?
3. Find all solutions for each value of k.

Problem 3 (Exercises 1.1.10, 1.2.9, 1.2.10, 1.2.13)**.** Solve each of the following linear systems by first writing the system's augmented matrix, then performing Gauss-Jordan elimination so that the matrix is in reduced row-echelon form (RREF), and then rewriting the system in terms of variables and equations. Finish your solution with a concluding sentence. You may use Sage to check your answers, but you must show each step in the elimination process.

1.

$$\begin{aligned} x + 2y + 3z &= 0 \\ 2x + 4y + 7z &= 2 \\ 3x + 7y + 11z &= 8 \end{aligned}$$

2.

$$\begin{aligned} x_4 + 2x_5 - x_6 &= 2 \\ x_1 + 2x_2 + x_5 - x_6 &= 0 \\ x_1 + 2x_2 + 2x_3 - x_5 + x_6 &= 2 \end{aligned}$$

3.

$$\begin{aligned} 4x_1 + 2x_2 + 2x_3 - x_4 &= 4 \\ 5x_1 + 4x_2 + 3x_3 - x_4 &= 5 \\ -2x_1 - 2x_2 - x_3 + 2x_4 &= -3 \\ 11x_1 + 6x_2 + 4x_3 + x_4 &= 11 \end{aligned}$$

4.

$$\begin{aligned} 3x + 11y + 19z &= -2 \\ 7x + 23y + 39z &= 10 \\ -4x - 3y - 2z &= 6 \end{aligned}$$

Problem 4 (Exercise 1.2.18)**.** Determine whether each of the following matrices below is in RREF. If a matrix is not in RREF, state why not.

1.

$$\begin{bmatrix} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

2.

$$\begin{bmatrix} 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3.

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

4.

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \end{bmatrix}$$

Problem 5 (Exercise 1.2.36)**.** The *dot product* of two vectors

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

in $\mathbb{R}^n$ is defined by

$$\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

Note that the dot product of two vectors is not another vector, but rather a number (ie. a scalar). We say that $\vec{x}$ and $\vec{y}$ are *perpendicular* if $\vec{x} \cdot \vec{y} = 0$. Find all vectors in $\mathbb{R}^3$ perpendicular to

$$\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}.$$

Draw a sketch.

Problem 6. Consider the vectors

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{w} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \in \mathbb{R}^2.$$

A **linear combination** of v and w is a vector of the form $x_1\vec{v} + x_2\vec{w}$ for some given scalars $x_1, x_2 \in \mathbb{R}$. For example, $\vec{v} + 3\vec{w}$ is a linear combination where the scalars are 1 and 3. We can compute the precise value of $\vec{v} + 3\vec{w}$ as follows:

$$\vec{v} + 3\vec{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}.$$

For each of the following linear combinations, make a sketch of $\vec{v}, \vec{w}$, and the linear combination on the same set of axes, and compute its precise value like the example above.

1. $2\vec{v} + \frac{1}{2}\vec{w}$,
2. $\vec{v} - \vec{w}$.

Problem 7. Let $\vec{v}$ and $\vec{w}$ be as in the previous problem and consider also the vector

$$\vec{b} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}.$$

Make a sketch of $\vec{v}, \vec{w}$, and $\vec{b}$ on the same set of axes.

1. Scaling (ie. stretching or compressing) and translating $\vec{v}$ and $\vec{w}$ as necessary draw a picture showing that $\vec{b}$ can be written as a linear combination of $\vec{v}$ and $\vec{w}$.
2. Find all solutions x_1, x_2 of the equation

$$\vec{b} = x_1\vec{v} + x_2\vec{w}$$

by rewriting the equation as a linear system and using Gauss-Jordan elimination.

How are these two questions related to each other?