Math 211, Spring 2017 — Homework 3

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Due February 17

Problem 1 (Exercises 2.1.43 a,b)**.**

1. Consider the vector $\vec{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$. Is the transformation $T(\vec{x}) = \vec{v} \cdot \vec{x}$ (the dot product with $\vec{v}$) from $\mathbb{R}^3$ to $\mathbb{R}$ linear? If so, find the matrix of T.
2. Consider an arbitrary vector $\vec{v}$ in $\mathbb{R}^3$. Is the transformation $T(\vec{x}) = \vec{v} \cdot \vec{x}$ linear? If so, find the matrix of T (in terms of the components of $\vec{v}$).

Hint: use Theorem 2.1.3 to show linearity and Theorem 2.1.2 to find the matrix of T.

Problem 2 (Exercise 2.1.45)**.** Consider two linear transformations $T : \mathbb{R}^m \to \mathbb{R}^p$ and $S : \mathbb{R}^p \to \mathbb{R}^n$. Show that the transformation $C : \mathbb{R}^m \to \mathbb{R}^n$ given by $C(\vec{x}) = S(T(\vec{x}))$ is linear as well.

Problem 3 (Exercise 2.2.2)**.** Find the matrix of a rotation through an angle of 60 degrees in the counter-clockwise direction.

Problem 4 (Exercise 2.2.5)**.** The matrix

$$\begin{bmatrix} -0.8 & -0.6 \\ 0.6 & -0.8 \end{bmatrix}$$

represents a rotation. Find the angle of rotation (in radians).

Problem 5 (Exercise 2.2.6)**.** Let L be the line in $\mathbb{R}^3$ that consists of all scalar multiples of the vector $\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$. (We say that L is **spanned** by $\vec{v}$.) Find the orthogonal projection of the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ onto L.

Problem 6 (Exercises 2.2.10, 11)**.** Let L be the line spanned by the vector $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$.

1. Find the matrix of the orthogonal projection onto L.
2. Find the matrix of the reflection about L.
3. Let $\vec{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and find both the orthogonal projection of $\vec{x}$ onto L and the reflection of $\vec{x}$ about L.

Problem 7 (Exercise 2.2.13)**.** Suppose a line L in $\mathbb{R}^2$ contains the unit vector

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

(Recall that a vector is a unit vector if its length is 1.) Find the matrix A of the linear transformation $T(\vec{x}) = \text{ref}_L(\vec{x})$. Give the entries of A in terms of u_1 and u_2. Show that A is of the form $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, where $a^2 + b^2 = 1$.

Problem 8 (Exercises 2.2.19, 20)**.** Find the matrices of the following linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$. Note that if you use Theorem 2.1.2 and can visualize the transformation, this won't involve much, if any, calculation.

1. The orthogonal projection onto the xy-plane.
2. The reflection about the xz-plane.
3. The rotation about the z-axis through an angle of $\pi/2$, counter-clockwise as viewed from the positive z-axis.

Problem 9 (Exercise 2.2.32)**.** Consider the rotation matrix $D = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ and the vector $\vec{v} = \begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix}$, where α and β are arbitrary angles.

1. Draw a sketch to explain why $D\vec{v} = \begin{bmatrix} \cos(\alpha+\beta) \\ \sin(\alpha+\beta) \end{bmatrix}$.
2. Compute $D\vec{v}$ using the dot product form of matrix-vector multiplication. Write a concluding sentence which explains that you've just proven the angle-sum identities for sine and cosine. Cool!!

$$\cos(\alpha+\beta) = \ldots, \quad \sin(\alpha+\beta) = \ldots.$$

Problem 10 (Exercise 2.2.42)**.** Let $T(\vec{x}) = \text{proj}_L(\vec{x})$ be the orthogonal projection onto a line L in $\mathbb{R}^2$. What is the relationship between $T(\vec{x})$ and $T(T(\vec{x}))$? Justify your answer by computing $P\vec{x}$ and $P(P\vec{x})$ for an arbitrary vector $\vec{x} \in \mathbb{R}^2$, where P is the matrix of T.