Linear algebra, Spring 2017 — Rank of a matrix

Class on February 2

Suppose we have a linear system with n equations and m unknowns. We've learned that this system can be expressed as the equation $A\vec{x} = \vec{b}$ or as an augmented matrix

$$\left[\begin{array}{ccc} A & \vdots & \vec{b} \end{array} \right]$$

where A is an $n \times m$ matrix called the coefficient matrix and $\vec{b} \in \mathbb{R}^n$.

Exercise 1. Suppose we are told that the linear system is inconsistent. What important feature should rref(A) have? Can you give an example of what rref(A) might look like if A is a 5×3 matrix? How many leading 1's can your example have? What can you say about rank(A)?

Solution 1. The important feature is that rref(A) should have a row of zeroes. This provides the chance for there to be a contradicting equation like $0 = 1$. Any RREF 3×5 matrix with a row of zeroes would work as an example; it could have as many as 3 leading 1's.

Key statement about rank: If $A\vec{x} = \vec{b}$ is inconsistent, then rank$(A) < n$.
Proof: Since the system is inconsistent, rref(A) has a row of zeroes. Therefore, not every row has a leading 1. Since A has n rows, this means rank$(A) < n$.

Exercise 2. Suppose we are told that the linear system has infinitely many solutions. What should rref(A) look like this time? Can you give an example of what rref(A) might look like if A is a 5×3 matrix? How many leading 1's can your example have? What can you say about rank(A)?

Solution 2. The important feature is that the system should have free variables. Any RREF 3×5 matrix with strictly fewer than 3 leading 1's would work as an example.

Key statement about rank: If $A\vec{x} = \vec{b}$ has infinitely many solutions, then rank$(A) < m$.
Proof: Since the system has free variables, not all of the variables can be leading variables. Since m is the total number of variables, rank$(A) < m$.

Exercise 3. Suppose we are told that the linear system has exactly one solution. What should rref(A) look like this time?

1. Can you give an example of what rref(A) might look like if A is a 5×3 matrix? How many leading 1's can your example have? What can you say about rank(A)?

2. Could A be a 3×5 matrix? Why or why not?

Solution 3. If the system has exactly one solution, we cannot have free variables, which means every column of rref(A) must have a leading 1.

1. Our example this time must look like

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

   Key statement about rank: If $A\vec{x} = \vec{b}$ has exactly one solution, then $\text{rank}(A) = m$.
   Proof: All m variables must be leading variables.

2. A could not be a 3×5 matrix and correspond to a system with exactly one solution. If it did, then $\text{rank}(A) = 5$ our previous key statement about rank. But this means that there are 5 leading 1's, which is impossible since the number of leading 1's cannot be more than 3 (the number of rows of A).

Summary. The following theorem summarizes the key statements that came about from the previous exercises.

Theorem. *Let A be an $n \times m$ matrix and consider a linear system whose coefficient matrix is A. The following statements hold:*

1. *If the system is inconsistent, then* $\text{rank}(A) < n$.

2. *If the system has infinitely many solutions, then* $\text{rank}(A) < m$.

3. *If the system has exactly one solution, then* $\text{rank}(A) = m$.

4. *If a system has exactly one solution, then there must be at least as many equations of variables (ie. $n \geq m$).*