Math 211, Spring 2017 — Exam 1 notes

Exam 1 on March 2

Remarks

Your first exam will be in class on Thursday, March 2. There will be a mix of computational questions and conceptual questions. There will likely be some True/False questions where you'll be asked to give a short justification. The main topics and sample problems below are meant to give you some practice and a broad overview of what the problems will be like, but they're not meant to be a comprehensive list. You should also look over past homework, quizzes, worksheets, and notes. The exam will cover material from the first two chapters that we've discussed in class. There have been a few stray topics (eg. shear transformations in section 2.2, multiplying block matrices in section 2.3) that we've not discussed and those won't appear on the exam. Transition matrices won't appear on the exam either. If you're looking for extra practice, particularly with conceptual questions, I highly recommend the True/False exercises at the ends of the chapters.

You'll be allowed to bring one **sheet of notes**, the size of a regular 8.5×11 inch sheet of paper. You may use both sides. No other materials, nor calculators, will be allowed.

Important definitions

- matrix, vector, $\mathbb{R}^n$, identity matrix, diagonal matrix, square matrix, invertible matrix, matrix addition, scalar multiplication
- dot product, matrix-vector product, matrix-matrix product, linear combination of vectors
- RREF, linear system, inconsistent system, augmented matrix, elementary row operations, rank of a matrix
- linear transformation (both original definition, and the equivalent parts a and b form)
- orthogonal projection, reflection, rotation of vectors

Main topics

- Find the solution set of a system using Gauss-Jordan elimination and write it in set notation
- For a linear system $A\vec{x} = \vec{b}$ understand what rref(A) or rank(A) tells you about the solution set
- Understand how to add, scalar multiply, and find the dot product of vectors; understand how to multiply a matrix with a vector in both ways we learned (via dot product of rows of the matrix with the vector, and via linear combination of the columns of the matrix)
- Determine whether a transformation is linear
- Find the matrix of a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ by finding the value of $T(\vec{e}_1), \ldots, T(\vec{e}_m)$.

- Know how to orthogonally project vectors onto a line, reflect vectors across a line, rotate vectors in $\mathbb{R}^2$, and know the matrices of these linear transformations
- Know how to multiply two matrices and the relationship between matrix multiplication and composition of linear transformations
- Know how to find the inverse of 2×2 matrices, the inverse of general $n \times n$ matrices, and how to find the inverse of the product AB of two invertible matrices
- Know the inverse of a rotation matrix or a reflection matrix by geometric thinking
- Know the following equivalent ways of what it means for an $n \times n$ matrix A to be invertible
  - there is a matrix X such that $AX = I_n$ and $XA = I_n$
  - the linear system $A\vec{x} = \vec{b}$ has a unique solution for every $\vec{b} \in \mathbb{R}^n$
  - the linear system $A\vec{x} = \vec{0}$ has a unique solution (ie. $\vec{x} = \vec{0}$ is the only solution)
  - $\text{rref}(A) = I_n$
  - $\text{rank}(A) = n$

Sample problems

1. Compute the following

   (a) $\begin{bmatrix} 2 & -1 & 3 & 4 \\ 8 & 4 & -2 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 7 \\ 1 \end{bmatrix}$

   (b) $(\vec{u} \cdot \vec{v})\,\vec{w}$ where $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -4 \\ 2 \\ 5 \end{bmatrix}$, and $\vec{w} = 2\vec{u}$

2. Compute the rank of the matrix $\begin{bmatrix} 1 & 2 & 1 & 1 \\ 3 & 1 & 4 & 2 \\ 4 & 8 & 4 & 4 \\ -2 & 1 & -4 & -1 \end{bmatrix}$

3. Solve the following linear systems using Gauss-Jordan elimination and express your answer in set notation

   (a)

$$\begin{aligned} x - 4y + 3z &= 2 \\ 2x + y + 9z &= 4 \\ 2x + 4y + 3z &= 2 \end{aligned}$$

   (b)

$$\begin{aligned} x_1 + 2x_2 + x_3 \phantom{{}+2x_4} &= 2 \\ 2x_1 + x_2 + 3x_3 + 2x_4 &= 1 \\ 2x_1 + 2x_2 + 3x_3 \phantom{{}+2x_4} &= 2 \end{aligned}$$

(c)

$$\begin{aligned} x_1 + 2x_2 + x_3 + x_4 &= 2 \\ 2x_1 + x_2 + 4x_3 + 2x_4 &= 4 \\ 4x_1 + 8x_2 + 4x_3 + 4x_4 &= 2 \\ 3x_1 + 3x_5 + 5x_3 + 3x_4 &= 5 \end{aligned}$$

4. Suppose the coefficient matrix of a linear system $A\vec{x} = \vec{b}$ is

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 3 & -5 & 13 \\ 1 & -2 & 5 \end{bmatrix}$$

How many solutions could the system have?

5. Is the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ a linear combination of the vectors $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$?

6. Suppose A is a 7×5 matrix and $\text{rank}(A) = 5$. What is the reduced row-echelon form of A?

7. The function $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x - 2y \\ x \\ z - y \end{bmatrix}$ is a linear transformation. Find the matrix of T.

8. Show that any vector in $\mathbb{R}^n$ is a linear combination of the vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$.

9. Let L be the line spanned by $\vec{w} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$ and let $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

   (a) Find the projection of $\vec{e}_1, \vec{e}_2, \vec{x}$ onto L.

   (b) Find the reflection of $\vec{x}$ across L.

10. Rotate the vector $\vec{x}$ from the previous problem by an angle $\theta = \pi/4$ in the counter-clockwise direction.

11. Compute the following matrix products if possible

   (a) $\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

   (c) $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

12. Decide if the matrix $A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & -1 & 0 & 0 \\ 2 & 2 & 5 & 4 \\ 0 & 3 & 0 & 1 \end{bmatrix}$ is invertible. If it is, find its inverse.

13. True/False (Chapter One Exercises 4, 7, 17, 21, 25, 31). Provide a short justification or computation with your answer.

   (a) A system of four linear equations in three unknowns is always inconsistent

(b) If the 4×4 matrix A has rank 4, then any linear system with coefficient matrix A will have a unique solution.

(c) rank $\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} = 2$

(d) If the system $A\vec{x} = \vec{b}$ has a unique solution, then A must be a square matrix.

(e) If $\vec{u}, \vec{v}$ and $\vec{w}$ are nonzero vectors in $\mathbb{R}^2$, then $\vec{v}$ must be a linear combination of $\vec{u}$ and $\vec{w}$.

(f) There exists a 4×3 matrix A of rank 3 such that $A \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \vec{0}$.

14. True/False (Chapter Two Exercises 6, 9, 10, 15, 17, 21, 22, 35). Provide a short justification or computation with your answer.

(a) The formula $(A^2)^{-1} = (A^{-1})^2$ holds for all invertible matrices A.

(b) If A is a 3×4 matrix and B is a 4×5 matrix, then AB is a 5×3 matrix.

(c) The function $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ 1 \end{bmatrix}$ is a linear transformation.

(d) The formula $\det(2A) = 2\det(A)$ holds for all 2×2 matrices A.

(e) The matrix $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ is invertible.

(f) There exists an invertible $n \times n$ matrix with two identical rows.

(g) If $A^2 = I_n$ then A must be invertible.

(h) If $A^{17} = I_2$ then the matrix A must be I_2.