Math 211, Spring 2017 — Exam 2 notes

Exam 2 on April 13

Remarks

Your second exam will be in class on Thursday, April 13. There will be a mix of computational questions and conceptual questions. There will likely be some True/False questions where you'll be asked to give a short justification. The main topics and sample problems below are meant to give you some practice and a broad overview of what the problems will be like, but they're not meant to be a comprehensive list. You should also look over past homework, quizzes, worksheets, and notes. The exam will focus on material from chapters 3 and 6 that we've discussed in class, but past material also remains relevant. There have been a few stray topics (eg. similar matrices in 3.4, patterns in section 6.1) that we've not discussed and those won't appear on the exam. If you're looking for extra practice, particularly with conceptual questions, I highly recommend the True/False exercises at the ends of the chapters.

You'll be allowed to bring one **sheet of notes**, the size of a regular 8.5×11 inch sheet of paper. You may use both sides. No other materials, nor calculators, will be allowed.

Important definitions

- image and kernel of a linear transformation
- subspace, linear independence, basis, dimension
- $\mathcal{B}$-coordinates, $\mathcal{B}$-matrix of a linear transformation
- determinant of a matrix, expansion by cofactors

Main topics

- Know how to check whether a set is a subspace by verifying the three defining properties
- Understand the notion of linear independence from its definition (which involves redundant vectors), by checking for a non-trivial relation, and via computation involving Gauss-Jordan elimination
- Know how to check whether a set of vectors is a basis for $\mathbb{R}^n$ using Gauss-Jordan elimination
- Know how to construct bases of the image and kernel of a linear transformation using the RREF of its standard matrix
- Know the rank-nullity theorem, its connection with the RREF of a matrix, and how it's used to give restrictions on the dimensions of the image and kernel of a linear transformation
- Understand the geometric meaning of $\mathcal{B}$-coordinates and how to convert between them and standard coordinates

- Know how to construct the $\mathcal{B}$-matrix of a transformation using the formula in Theorem 3.4.3 and via the formula $A = SBS^{-1}$, and how to recall this idea using a commutative diagram
- For geometrically defined linear transformations (ie. reflections and projections), know how to construct a basis that makes the $\mathcal{B}$-matrix diagonal
- Understand the determinant as a tool to test for invertibility of a matrix
- Know how to compute determinants using Gauss-Jordan elimination, cofactor expansion, and using facts such as "the determinant of an upper triangular matrix is the product of its diagonal entries."
- Know other common facts about determinant such as
  - swapping rows/columns changes the sign of the determinant
  - scaling a row/column by $1/k$ changes the determinant by a factor of k
  - if a matrix has a row/column of 0's, its determinant is 0
  - if there is a linear relation among the rows/columns of a matrix, its determinant is 0
  - $\det A = \det A^T$
  - $\det(AB) = \det A \det B$
  - $\det A^{-1} = \frac{1}{\det A}$ when A is invertible

Sample problems

1. For each of the following subsets of $\mathbb{R}^4$ determine whether it is a subspace
   (a) $V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} : x_1 = x_3 - x_2, x_4 = 0 \right\}$
   (b) $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} : x_1 = x_3 - x_2, x_4 = x_1 x_3 \right\}$
2. Find the image and kernel of each of the following linear transformations and give a brief justification
   (a) Reflection about a plane in $\mathbb{R}^3$
   (b) Projection onto the line $y = x$ in $\mathbb{R}^2$
   (c) $T(\vec{x}) = A\vec{x}$ where A is an $n \times m$ matrix with rank m
3. Determine whether the following vectors are linearly independent
$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 7 \\ 10 \end{bmatrix}$$
4. Find a basis for the kernel and image of the following matrix
$$A = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ -3 & 0 & 0 & 1 & 0 \\ -1 & 0 & 4 & 1 & 2 \end{bmatrix}.$$

5. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bases of $\mathbb{R}^2$ consisting of

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \text{and} \quad \vec{w}_1 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}, \quad \vec{w}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

respectively.

(a) Suppose $\vec{x} \in \mathbb{R}^2$ is a vector with $\mathcal{B}_1$ coordinates $\begin{bmatrix} 5 \\ 5 \end{bmatrix}$. Find $\vec{x}$ in standard coordinates.

(b) Suppose $\vec{y} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ in standard coordinates. Find the $\mathcal{B}_2$ coordinates of $\vec{y}$.

(c) Suppose $\vec{z}$ has $\mathcal{B}_1$ coordinates $\begin{bmatrix} -1 \\ -2 \end{bmatrix}$. Find the $\mathcal{B}_2$ coordinates of $\vec{z}$.

6. For each of the following linear transformations $T : \mathbb{R}^3 \to \mathbb{R}^3$ find a basis $\mathcal{B}$ of $\mathbb{R}^3$ which makes the $\mathcal{B}$-matrix of T diagonal. Give the $\mathcal{B}$-matrix for each too.

(a) T is reflection across the line spanned by $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(b) T is reflection across the plane V given by the equation $x_1 + 2x_2 + 3x_3 = 0$.

(c) T is orthogonal projection onto the plane V in the previous part.

7. Find the determinant of the following matrix

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 & 1 \\ 1 & 2 & 3 & 2 & 1 \\ 0 & 4 & 4 & 5 & 1 \\ 1 & 2 & 4 & 2 & 2 \\ 0 & 1 & 1 & 2 & 1 \end{bmatrix}.$$

8. True/False:

(a) If A is a 5×6 matrix of rank 4, then the kernel of A contains two linearly independent vectors.

(b) If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ are linearly independent, then $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ must be linearly independent as well.

(c) If $2\vec{u} + 3\vec{v} + 4\vec{w} = 5\vec{u} + 6\vec{v} + 7\vec{w}$, then $\{\vec{u}, \vec{v}, \vec{w}\}$ must be linearly dependent.

(d) The column vectors of a 5×4 matrix must be linearly dependent

(e) The column vectors of a 4×5 matrix must be linearly dependent

9. True/False

(a) $\det(A^{10}) = (\det A)^{10}$ for all 10×10 matrices A

(b) $\det(4A) = 4 \det A$ for all 4×4 matrices A

(c) $\det \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = 1$

(d) If the determinant of a 4×4 matrix is 4, then its rank must be 4

(e) There exists a nonzero 4×4 matrix such that $\det A = \det(4A)$

(f) If A and B are $n \times n$ matrices such that $A = SBS^{-1}$ for some $n \times n$ invertible matrix S then $\det A = \det B$

Solutions

The following are answers to the True/False questions at the end of the chapters.

- Chapter 3: TFFTT FTFTF TFTTT TTTFT TTFFT TTTFF FTTFF FFTTF TFTTT TTFTT FTF
- Chapter 6: TTTTT FFTFF TFTFT TTFTF FTTTF FTFFT FFFFF TTTTF TFTTT TF

Answers to the sample problems are below.

Answers to practice problems for exam 2.

1a. Yes, it is a subspace.

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in V \quad \checkmark$$

Let $\begin{bmatrix} x_3 - x_2 \\ x_2 \\ x_3 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} y_3 - y_2 \\ y_2 \\ y_3 \\ 0 \end{bmatrix}$ be in V.

Then
$$\begin{bmatrix} x_3 - x_2 \\ x_2 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} y_3 - y_2 \\ y_2 \\ y_3 \\ 0 \end{bmatrix} = \begin{bmatrix} x_3 + y_3 - (x_2 + y_2) \\ x_2 + y_2 \\ x_3 + y_3 \\ 0 \end{bmatrix}$$

is in V $\checkmark$.

And if k is a scalar then

$$k \begin{bmatrix} x_3 - x_2 \\ x_2 \\ x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} kx_3 - kx_2 \\ kx_2 \\ kx_3 \\ 0 \end{bmatrix}$$

is in V. $\checkmark$.

1b. No, it is not a subspace.

$\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ is in W

$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}$ is in W

but $\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \\ 3 \end{bmatrix}$ is NOT in W

because $x_4 \neq x_1 x_3$.

2a. Image is all of $\mathbb{R}^3$
Kernel is $\{\vec{0}\}$.

b. Image is the line $y=x$.
Kernel is the line through the origin orthogonal to $y=x$.

c. Image is a subspace of dimension m.
Kernel is $\{\vec{0}\}$.

3. Not linearly indep.

$$2\begin{bmatrix}1\\1\\1\\1\end{bmatrix} - 3\begin{bmatrix}1\\2\\3\\4\end{bmatrix} + 1\begin{bmatrix}1\\4\\7\\10\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}.$$

alternatively, $$\mathrm{rref}\begin{bmatrix}1&1&1\\1&2&4\\1&3&7\\1&4&10\end{bmatrix} = \begin{bmatrix}1&0&-2\\0&1&3\\0&0&0\\0&0&0\end{bmatrix}$$

4. $\operatorname{rref} A = \begin{bmatrix} 1 & 0 & 0 & -1/3 & 0 \\ 0 & 0 & 1 & 1/6 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Basis of $\operatorname{Im} A$ is $\left(\begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} \right)$.

Basis of $\ker A$ is $\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 0 \\ -1/6 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1/2 \\ 0 \\ 1 \end{bmatrix} \right)$.

5. $\mathcal{B}_1 = (\vec{v}_1, \vec{v}_2)$. $\mathcal{B}_2 = (\vec{w}_1, \vec{w}_2)$.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \vec{w}_1 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}, \quad \vec{w}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

a. $[\vec{x}]_{\mathcal{B}_1} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$

$$\vec{x} = 5\vec{v}_1 + 5\vec{v}_2 = 5\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 5\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \boxed{\begin{bmatrix} 15 \\ 25 \end{bmatrix}.}$$ (standard coordinates).

b. $\vec{y} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ (standard coordinates).

$$\vec{y} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} = c_1\vec{w}_1 + c_2\vec{w}_2 = c_1\begin{bmatrix} -3 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

so

$$\left.\begin{aligned} -3c_1 + c_2 &= 1 \\ 2c_1 - c_2 &= 4 \end{aligned}\right| \xrightarrow{\text{solve}} \begin{aligned} c_1 &= -5 \\ c_2 &= -14 \end{aligned}$$

so $[\vec{y}]_{\mathcal{B}_2} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \boxed{\begin{bmatrix} -5 \\ -14 \end{bmatrix}}.$

S. c. $[\vec{z}]_{\mathcal{B}_1} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ So $\vec{z} = -1\,\vec{v}_1 - 2\,\vec{v}_2 = -1\begin{bmatrix} 1 \\ 3 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ -7 \end{bmatrix}.$

$$\vec{z} = c_1\vec{w}_1 + c_2\vec{w}_2 = c_1\begin{bmatrix} -3 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -5 \\ -7 \end{bmatrix}.$$

so
$$\begin{aligned} -3c_1 + c_2 &= -5 \\ 2c_1 - c_2 &= -7. \end{aligned}$$

$\xrightarrow{\text{solve}}$ $c_1 = 12$, $c_2 = 31$

$$\boxed{[\vec{z}]_{\mathcal{B}_2} = \begin{bmatrix} 12 \\ 31 \end{bmatrix}.}$$

6\. a. choose one vector parallel to $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

and two lin. indep vectors orthogonal to $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

$$\boxed{\mathcal{B} = \left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right)}$$ is one basis that

makes the $\mathcal{B}$matrix of T diagonal.

$$\boxed{B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}$$

b. choose two linearly independent vectors in the plane $x_1 + 2x_2 + 3x_3 = 0$

and one vector orthogonal to the plane. Ex:

$$\mathcal{B} = \left(\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right)$$ works.

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

6c. The same basis works for this problem.

$$\boxed{\mathcal{B} = \left(\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right)}$$

$$\boxed{B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.}$$

7. $\det A = 0$.

8\. a. True $(\dim(\ker A) = 2)$.

b. True.

c. True.

d. False. (ex. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$)

e. True. (can have at most 4 lin. indep vectors in $\mathbb{R}^4$).

9a. true.

b. False. (ex. $\cancel{\det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 \text{ but } \det \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = 16.}$).

$$\det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 1, \text{ but } \det \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = 256$$

c. ~~True~~

False (cofactor expansion along last row, or you can see it by doing 3 row operations to get I_4).

9d.

~~9a~~. True. (It is invertible, so its rank must be 4).

e. ~~False!~~ True ($\det 4A = 4^4 \det A$, ~~[illegible]~~ see Theorem 6.2.3. so whenever $\det A = 0$ $\det A = \det 4A$.)

f. True. $\left(\det A = \det(SBS^{-1}) = \det S \cdot \det B \cdot \det S^{-1} = \det S \cdot \frac{1}{\det S} \cdot \det B = \det B. \right)$.

see Theorem 6.2.6 and 6.2.8.