Math 211, Spring 2017 — Final exam notes

Final Exam self-scheduled May 4-8

Remarks

Your final exam will be self-scheduled during the final exam period, which runs May 4 to May 8. A few important reminders about the exam period:

- You may only take one exam during each 2 hour exam session.
- There is only one exam session on the final day, from 9:00 to 11:30 am.
- Exams are picked up from Kendade Atrium for the May 4, 7:00 pm session, and from Hooker Auditorium for all other sessions.
- More information is posted on the registrar's web page.

The exam will be cumulative with an emphasis on material covered since the last exam, including material from section 5.1. Since you won't have had homework on Chapter 5 material, any questions from this will be straightforward and quite similar to sample problems below. The material below discusses topics since the last exam; please refer to the previous review guides for summaries of older material.

There will be a mix of computational questions and conceptual questions. It will be roughly two questions longer than our in-class exams. There will likely be some True/False questions and some short answer questions where you'll be asked to give short justifications. The sample problems below are meant to give you some practice and a broad overview of what the problems will be like, but they're not meant to be a comprehensive list.

You should also look over past homework, quizzes, worksheets, and notes. There have been a few stray textbook topics that we've not discussed and those won't appear on the exam. If you're looking for extra practice, particularly with conceptual questions, I highly recommend the True/False exercises at the ends of the chapters.

You'll be allowed to bring one **sheet of notes**, the size of a regular 8.5×11 inch sheet of paper. You may use both sides. You **must turn in your sheet of notes with your exam packet**. For those who would like to decorate their sheets or otherwise artistically arrange them: I will post some on the class web page after the exam (but please let me know if you'd prefer that I not post yours). No other materials, nor calculators, will be allowed.

Important definitions

- eigenvalue, eigenvector, characteristic polynomial, characteristic equation
- eigenspace, basis for eigenspace, eigenbasis for $\mathbb{R}^n$
- algebraic and geometric multiplicity
- orthonormal basis, orthogonal projection onto a subspace, orthogonal complement of a subspace, angle between two vectors

Main topics

- understand the eigenvalues and eigenvectors for common geometrically defined linear transformations (reflections, rotations, orthogonal projections)
- know how to find eigenvalues of a matrix by finding roots of the characteristic polynomial
- know how to find basis for an eigenspace by finding a basis for $\ker(A - \lambda I)$
- know what makes a matrix diagonalizeable (it has to yield an eigenbasis) and how to diagonalize such matrices
- know how to compute $A^t\vec{x}$ for arbitrary t when A is diagonalizeable
- understand, using eigenvalues and eigenvectors, how different initial conditions $\vec{x}(0)$ affect $\vec{x}(t)$ for systems defined by $\vec{x}(t+1) = A\vec{x}(t)$.
- orthonormal sets of vectors are linearly independent, computing the orthogonal projection of a vector onto a subspace, Pythagorean theorem, Cauchy-Schwarz inequality

Sample problems

1. Fill in the blank: suppose that A is the standard matrix of a linear transformation $T : \mathbb{R}^9 \to \mathbb{R}^9$. If the eigenvalues of A are $\lambda_1, \lambda_2, \ldots, \lambda_p$ with corresponding eigenspaces $E_{\lambda_1}, E_{\lambda_2}, \ldots, E_{\lambda_p}$, then A is diagonalizeable if and only if

$$\dim(E_{\lambda_1}) + \dim(E_{\lambda_2}) + \cdots + \dim(E_{\lambda_p}) = \underline{\hspace{3cm}}.$$

2. Let $A = \begin{bmatrix} 2 & 0 & 2 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$. Find all eigenvalues of A and a basis for each eigenspace. Is A diagonalizeable? If so, find an invertible matrix S and a diagonal matrix B such that $A = SBS^{-1}$.

3. Find a closed formula for A^t where t is an arbitrary postive integer and A is the matrix from the previous exercise.

4. Suppose two interacting populations of coyotes and roadrunners can be modeled by the recursive equations

$$\begin{aligned} c(t+1) &= \phantom{-1.5c(t) + {}} 0.75r(t) \\ r(t+1) &= -1.5c(t) + 2.25r(t). \end{aligned}$$

Give examples of initial population sizes $c(0) \neq 0$ and $r(0) \neq 0$ that guarantee both populations will become extinct in the long run.

5. Find the orthogonal projection of $9\vec{e}_1 \in \mathbb{R}^4$ onto the subspace $V \subseteq \mathbb{R}^4$ spanned by

$$\begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -2 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

6. Consider orthonormal vectors $\vec{u}_1, \ldots, \vec{u}_5 \in \mathbb{R}^{10}$. Find the length of the vector

$$\vec{x} = 7\vec{u}_1 - 3\vec{u}_2 + 2\vec{u}_3 + \vec{u}_4 - \vec{u}_5.$$

Hint: use the Pythagorean theorem.

7. Suppose

$$W = \operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} \right).$$

What is the dimension of $W^{\perp}$? Explain why the kernel of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$

is a 2-dimensional subspace of $\mathbb{R}^4$ without doing any computation.

8. Suppose $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^4$ are vectors such that

$$\vec{v}_1 \cdot \vec{v}_1 = 3, \quad \vec{v}_2 \cdot \vec{v}_2 = 9, \quad \vec{v}_3 \cdot \vec{v}_3 = 49$$

and

$$\vec{v}_1 \cdot \vec{v}_2 = 5, \quad \vec{v}_1 \cdot \vec{v}_3 = 11, \quad \vec{v}_2 \cdot \vec{v}_3 = 20.$$

(a) Find $\|\vec{v}_2\|$.

(b) Find the angle enclosed by the vectors $\vec{v}_2$ and $\vec{v}_3$.

(c) Find $\|\vec{v}_1 + \vec{v}_2\|$. Hint: try finding $\|\vec{v}_1 + \vec{v}_2\|^2$ first.

(d) Find $\operatorname{proj}_{\vec{v}_2}(\vec{v}_1)$, expressed as a scalar multiple of $\vec{v}_2$.

9. True/false:

(a) If $\vec{x}, \vec{y} \in \mathbb{R}^n$ then the equation $\|\vec{x} + \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$ must hold.

(b) There exists a subspace $V \subseteq \mathbb{R}^5$ such that $\dim V = \dim V^{\perp}$.

(c) If V is a subspace of $\mathbb{R}^n$ and $\vec{x}$ is a vector in $\mathbb{R}^n$, then the vector $\operatorname{proj}_V(\vec{x})$ must orthogonal to $\vec{x} - \operatorname{proj}_V(\vec{x})$.

10. True/False:

(a) If 0 is an eigenvalue of A then $\det A = 0$

(b) The eigenvalues of any lower triangular matrix are its diagonal entries.

(c) If $\vec{v}$ is an eigenvector of A then it must be an eigenvector of A^3 as well.

(d) All invertible matrices are diagonalizeable. Hint: consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

(e) If an invertible matrix A is diagonalizeable, then A^{-1} must be diagonalizeable as well.