Section 11.1 What is a Differential Equation?

Example 1 Suppose a ball is dropped from the top of a building of height 50 meters. Let $h(t)$ denote the height of the ball after $t$ seconds, then it is known that $h$ satisfies the following three conditions:

$$h''(t) = -9.8, \quad h'(0) = 0, \quad h(0) = 50.$$ 

Find a formula for $h(t)$.

Definition A differential equation is an equation involving derivatives of a function. To solve a differential equation, one needs to find a function (or functions) that satisfies the given equation.

Example 2 For a differential equation $\frac{dy}{dx} = 2y$, $y = e^{2x}$ is a solution.

Example 3 For a differential equation $\frac{d^2y}{dx^2} = -y$, $y = \cos x$ is a solution. $y = \sin x$ is also a solution.

Example 4 Determine a constant $k$ so that $y = x^3 + k$ is a solution to the differential equation

$$xy' - 3y = 30.$$
Definition We say that a differential equation is first-order if the equation involves the first derivative, but no higher derivatives. Similarly, a differential equation is second-order if the equation contains the second derivative, but no higher derivatives.

Example 5 \( \frac{dy}{dx} = 2y \) is first-order, and \( \frac{d^2y}{dx^2} = -y \) is second-order.

Example 6 Solve the following differential equations:

a. \( y' = 3x^2 \).

b. \( \frac{dy}{dx} = \frac{1}{x} \).

c. \( y'' = e^x \).

d. \( \frac{d^2y}{dx^2} = \sin x \).

Remark In general, the solutions to an \( n \)th-order differential equation contains \( n \) arbitrary constants. Therefore, to find a specific solution to an \( n \)th-order differential equation, there must be \( n \) initial conditions.

Example 7 Solve the equations above with the following initial conditions

a. \( y' = 3x^2 \), \( y(0) = 3 \).

b. \( \frac{dy}{dx} = \frac{1}{x} \), \( y(e) = 2 \).

c. \( y'' = e^x \), \( y(0) = 2 \), \( y'(0) = 3 \).

d. \( \frac{d^2y}{dx^2} = \sin x \), \( y(0) = \frac{dy}{dx}(0) = 1 \).
Section 11.2 Slope Fields

In this section, we will study how to visualize solutions of a first-order differential equation.

Example 1 Consider the equation \( \frac{dy}{dx} = y \) and its slope field.

Example 2 Considering the equation \( \frac{dy}{dx} = 2x \) and its slope field, visualize the solution to the initial value problem

\[
\frac{dy}{dx} = 2x, \quad y(1) = 2.
\]
Example 3 Draw the slope field of each of the following differential equations and solve given initial value problems:

a. $\frac{dy}{dx} = 2 - y$
   i. $y = 1$ when $x = 0$
   ii. $y = 0$ when $x = 1$
   iii. $y = 3$ when $x = 0$

b. $\frac{dy}{dx} = \frac{x}{y}$
   i. $y = 1$ when $x = 0$
   ii. $y = 0.5$ when $x = 1$
   iii. $y = 3$ when $x = 0$

Remark The webpage
Section 11.3 Euler’s Method

In the previous section, we learned how to visualize solutions of a given differential equation using the slope field of the equation. In this section, we develop a method to estimate the solution numerically. This method is called Euler’s method.

Motivation If $\Delta x$ is small, then $f(a + \Delta x) \approx f'(a)\Delta x + f(a)$.

Example 1 Given $\frac{dy}{dx} = y$, $y(0) = 1$, estimate $y(1)$ using Euler’s method with $\Delta x = 0.2$.

Example 2 Given $\frac{du}{dx} = 2x$, $y(0) = 1$, estimate $y(1)$ using Euler’s method with $\Delta x = 0.2$. Compare your result with the exact value.
Section 11.4 Separation of Variables

In Sections 11.2 and 11.3, we used slope field and Euler’s method to approximate solutions. In this section, we study how to solve a differential equation analytically (i.e., how to find a formula for the solution(s)).

Example 1 Solve \( \frac{dy}{dx} = 3y \).

Example 2 Solve \( \frac{dy}{dx} = -y + 20 \).

Example 3 Solve \( \frac{dy}{dx} = 2y - 2yx \).
Example 4 Solve \( \frac{du}{dt} = \frac{1}{2} u^2 \), \( u(0) = 1 \).

Example 5 Solve \( \frac{dy}{dt} = y(2 - y) \), \( y(0) = 1 \).
Section 11.5 Growth and Decay

Consider the differential equation
\[ \frac{dP}{dt} = kP, \]
where \( k \) is a given constant. This equation can be easily solved and every solution to this equation must be in the form
\[ P(t) = Ce^{kt}, \]
where \( C \) is a constant.

Example 1 A bank account earns interest continuously at a rate of 5% of the current balance per year. Assume that the initial deposit is $1,000, and that no other deposits or withdrawals are made, then it is known that \( B(t) \), the balance after \( t \) years, satisfies the differential equation
\[ \frac{dB}{dt} = 0.05B. \]

a. Solve the differential equation.

b. How long does it take for the money to double?

Example 2 When a patient is given a certain antibiotic, as the drug passes through the liver and kidneys, some portion of the drug is eliminated every hour and it is known that \( Q(t) \), the quantity of the antibiotic (in mg) that remains inside body after \( t \) hours, satisfies the differential equation
\[ \frac{dQ}{dt} = -0.51Q. \]

Assume that 250 mg of the antibiotic was given initially and find \( Q(t) \).
**Example 3** When a murder is committed in a room where the temperature of the surrounding air is a constant 20°C, the body, originally at 37°C, cools according to Newton’s Law of Cooling, which states that \( H(t) \), the temperature of the body, satisfies the differential equation

\[
\frac{dH}{dt} = -k(H - 20)
\]

for some positive constant \( k \). Suppose that after two hours the temperature of the body is 35°C. If the body is found at 4 pm at a temperature of 30°C, when was the murder committed?

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**Radioactive Dating** Half-life is the period of time it takes for a substance undergoing decay to decrease by half. For example, the half-life of \( ^{14}\text{C} \) (carbon 14) is 5,730 years.

**Example 4** Let \( C(t) \) (in grams) be the amount of \( ^{14}\text{C} \) in a certain material at time \( t \) (in years). If \( C(0) = 6 \), then \( C(5,730) = \) and \( C(11,460) = \). It is known that the rate of change of \( C(t) \) is negatively proportional to \( C(t) \), that is, there is a negative constant \( k \) such that \( \frac{dC}{dt} = kC \). Find \( k \).

**Example 5** In 1988, three universities in the world dated a piece of the *Shroud of Turin* that is made of cotton. In all three independent experiments, the amount of \( ^{14}\text{C} \) in the cloth was about 92% of the amount in living plants today. The amount of \( ^{14}\text{C} \) in living plants today is assumed to be the same as the amount of \( ^{14}\text{C} \) in living plants when the cloth was made. Estimate the age of the cloth.
Think about the velocity of a sky-diver jumping out of a plane. When the sky-diver first jumps, his velocity is zero. The pull of gravity then makes his velocity increase. As the sky-diver speeds up, the air resistance also increases. Since the air resistance partly balances the pull of gravity, the force causing him to accelerate decreases. Thus, the velocity is an increasing function of time, but it is concave down.

**Example 1** Let \( v(t) \) denote the velocity of the sky-diver. We only consider the forces due to gravity and air resistance. Assume that the air resistance is proportional to the velocity of the sky-diver, so the total net force would be given by

\[
F = mg - kv,
\]

where \( m \) is the mass of the sky-diver, \( g \) is the gravitational acceleration, and \( k \) is a positive constant. By Newton’s Second Law of Motion, we know that

\[
\text{Net Force} = \text{Mass} \times \text{Acceleration}
\]

which in our case becomes

\[
mg - kv = m \frac{dv}{dt}.
\]

Solve this differential equation and compute the limit of the solution as \( t \to \infty \).
Section 11.7 Models of Population Growth

In this section, we study two important models of population growth. We begin with the exponential model.

Example 1 In the early 19th century, an English scholar Thomas Malthus came up with the exponential model of population growth. According to his theory, the rate of change in the population is proportional to the population itself. Let $k$ be the constant of proportionality. Set up a differential equation that governs the population and solve it.

Example 2 In 1838, a French scholar Pierre-François Verhulst suggested to use a different model for population growth. He claimed that the population function satisfies the differential equation

$$\frac{dP}{dt} = kP \left(1 - \frac{P}{L}\right),$$

where $k$ and $L$ are constants. Solve this differential equation.
Section 11.11  Linear Second-Order Differential Equations

In this section, we will study how to solve second-order differential equations of the form

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0,$$

where $a, b,$ and $c$ are constants with $a \neq 0$.

**Example 1** Recall how we defined complex exponents:

$$e^{it} = \cos t + i \sin t.$$  

In general,

$$e^{s+it} = e^s e^{it} = e^s(\cos t + i \sin t).$$

**Example 2** Show that $y = e^x$ and $y = e^{-2x}$ are solutions to $\frac{d^2 y}{dx^2} + \frac{dy}{dx} - 2y = 0$.

**Example 3** Show that $y = 2e^x - 5e^{-2x}$ is also a solution to $\frac{d^2 y}{dx^2} + \frac{dy}{dx} - 2y = 0$.

**Principle of Superposition** Suppose that $f_1(x)$ and $f_2(x)$ are solutions to $a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0$. Then so is $C_1 f_1(x) + C_2 f_2(x)$ for any constants $C_1$ and $C_2$.

**Proof**
Definition Let a differential equation $a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0$ be given. The algebraic equation $ar^2 + br + c = 0$ is called the characteristic equation of the differential equation.

Example 4 Find and solve the characteristic equation of $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$.

Theorem 1 Let a differential equation $a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0$ be given. If the characteristic equation $ar^2 + br + c = 0$ has two distinct real roots, say $r_1$ and $r_2$, then the general solution of $a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0$ is given by $y = C_1 e^{r_1x} + C_2 e^{r_2x}$. 
Example 5 Solve

a. \( \frac{d^2 y}{dx^2} + 3 \frac{dy}{dx} - 10y = 0. \)

b. \( y'' - 4y = 0. \)

c. \( \frac{d^2 y}{dx^2} - 5 \frac{dy}{dx} = 0. \)

d. \( y'' - y' - 3y = 0. \)

Example 6 Solve

a. \( \frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + 3y = 0, \ y(0) = 7, \ y'(0) = 11. \)

b. \( y'' + 3y' + 2y = 0, \ y(0) = -\frac{1}{2}, \ y'(0) = 3. \)
**Theorem 2** Let a differential equation $a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0$ be given. If the characteristic equation $ar^2 + br + c = 0$ has *only one root*, say $r$, then the general solution of $a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0$ is given by $y = (C_1 + C_2x)e^{rx}$.

**Example 7** Solve

a. $\frac{d^2y}{dx^2} - 4 \frac{dy}{dx} + 4y = 0.$

b. $y'' + 6y' + 9y = 0,\; y(0) = 1,\; y'(0) = 2$. 
Theorem 3 Let a differential equation \( a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0 \) be given. If the characteristic equation \( ar^2 + br + c = 0 \) has two complex roots, say \( \alpha \pm i\beta \), then the general solution of \( a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0 \) is given by \( y = C_1 e^{\alpha x} \cos \beta x + C_2 e^{\alpha x} \sin \beta x \).

Example 8 Solve

a. \( \frac{d^2 y}{dx^2} + 9y = 0 \). 

b. \( y'' + 4y' + 13y = 0, \ y(0) = 0, \ y'(0) = 30 \).
Consider a body of mass $m$ attached to the end of a spring. The other end of the spring is attached to a fixed wall. Assume that the body rests on a horizontal plane without friction, so the body moves only back and forth as the spring compresses and stretches. Denote by $x$ the distance of the body from its equilibrium position - its position when the spring is unstretched. We take $x > 0$ when the spring is stretched, and $x < 0$ when it is compressed.

According to Hooke’s law, the restorative force $F_S$ that the spring exerts on the body is proportional to the distance $x$ that the spring has been stretched or compressed. Let $k$ denote the constant of proportionality ($k$ is a characteristic of the spring), then it follows that

$$F_S = -kx.$$  

By Newton’s Second Law of Motion, we have

$$m \frac{d^2x}{dt^2} = F_S = -kx$$

or

$$m \frac{d^2x}{dt^2} + kx = 0.$$  

Now we also suppose that the body is attached to a dashpot. The dashpot provides a force directed opposite to the direction of the motion of the body. We assume that the dashpot is so designed that this force $F_D$ is proportional to the velocity $v = \frac{dx}{dt}$ of the body. If $c$ denotes the constant of proportionality ($c$ is a characteristic of the dashpot), then it follows that

$$F_D = -cv = -c \frac{dx}{dt}.$$  

Again by Newton’s Second Law of Motion, we have

$$m \frac{d^2x}{dt^2} = F_S + F_D = -kx - c \frac{dx}{dt}$$

or

$$m \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx = 0.$$  

This equation is called the spring equation.

Remark

a. The constant $k$ is called the spring constant and measures the stiffness of the spring.

b. The constant $c$ is called the dashpot constant and measures the viscosity of the dashpot.
Example 1 A body with mass \( m = \frac{1}{2} \) kilogram (kg) is attached to the end of a spring that is stretched 2 meters (m) by a force 100 newtons (N). It is set in motion with initial position \( x_0 = 0.5 \) (m) and initial velocity \( v_0 = -10 \) (m/s).

a. Find the spring constant \( k \).

b. Find the position function of the body.

Example 2 The mass and spring of Example 1 are now attached also to a dashpot that provides 6 N of resistance for each meter per second of velocity. The mass is set in motion with the same initial position \( x_0 = 0.5 \) (m) and the same initial velocity \( v_0 = -10 \) (m/s). Find the position function of the body.
Definition Consider the spring equation \( m \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx = 0 \).

a. If \( c = 0 \), then the equation is said to be undamped.

b. When \( c > 0 \),
   
   (a) If \( c^2 < 4km \), then the equation is said to be underdamped.
   
   (b) If \( c^2 = 4km \), then the equation is said to be critically damped.
   
   (c) If \( c^2 > 4km \), then the equation is said to be overdamped.

Example 3 Discuss the behavior of the position function when the spring equation is

a. undamped

b. underdamped

c. critically damped

d. overdamped
Section 11.8 Systems of Differential Equations

In this section, we will study how to solve a certain type of systems of differential equations. Let \( x = x(t) \) and \( y = y(t) \) be functions of \( t \), so \( x' = x'(t) = \frac{dx}{dt} \) and \( y' = y'(t) = \frac{dy}{dt} \).

**Example 1** Solve the system, that is, find \( x(t) \) and \( y(t) \) that satisfy

\[
\begin{align*}
x' &= -2y \\
y' &= \frac{1}{2}x.
\end{align*}
\]

**Example 2** Solve the system

\[
\begin{align*}
x' &= -2y \\
y' &= \frac{1}{2}x
\end{align*}
\]

with initial conditions \( x(0) = 2, \ y(0) = 0 \).

**Example 3** Solve the system

\[
\begin{align*}
x' &= y \\
y' &= 2x + y.
\end{align*}
\]
Example 4 Solve the system

\[
\begin{align*}
  x' &= 4x - 3y \\
  y' &= 6x - 7y.
\end{align*}
\]

with initial conditions \( x(0) = 2, \ y(0) = -1 \).

Example 5 Consider a conflict between two armies of \( x \) and \( y \) soldiers, respectively. An English engineer F. W. Lanchester assumed that if both armies are fighting a conventional battle within sight of one another, the rate at which soldiers in one army are put out of action (killed or wounded) is proportional to the number of soldiers in the opposing army. This assumption therefore leads to a system of differential equations

\[
\begin{align*}
  \frac{dx}{dt} &= -ay \\
  \frac{dy}{dt} &= -bx, \quad a, b > 0
\end{align*}
\]