Section 9.1 Sequences

Definition A sequence is an infinite list of numbers \(s_1, s_2, s_3, \ldots, s_n, \ldots\) We call \(s_1\) the first term, \(s_2\) the second term, and so on. \(s_n\) is the general term, say \(n^{th}\) term.

Example 1 1, 4, 9, 16, \ldots is a sequence. To be more precise, \(s_1 = 1, s_2 = 4, s_3 = 9, \) and so on. In this example, the general \(n^{th}\) term is given by \(s_n = \ldots\).

Example 2 Give the first six terms of the following sequences:

a. \(s_n = \frac{n(n+1)}{2}\)

b. \(t_n = \frac{n+(-1)^n}{n}\)

Example 3 Consider the sequence 1, 1, 2, 3, 5, 8, 13, \ldots Can you read off the pattern? What is the next term?

Example 4 Find the general \(n^{th}\) term \(s_n\) when the first few terms of the sequence are

a. 2, 4, 6, 8, 10, 12, \ldots

b. 1, 3, 5, 7, 9, 11, \ldots

c. 1, 2, 4, 8, 16, 32, \ldots

d. 1, -1, 1, -1, 1, -1, \ldots

e. \(\frac{1}{2}, \frac{-2}{3}, \frac{3}{4}, \frac{-4}{5}, \frac{5}{6}, \frac{-6}{7}, \ldots\)

Sequences can also be defined recursively, by giving an equation relating the \(n^{th}\) term to the previous terms (called recursion formula) and as many of the first terms as are needed to get started.

Example 5 Give the first few terms of the following recursively defined sequences.

a. \(s_n = s_{n-1} + 3\) for \(n > 1\) and \(s_1 = 4\).

b. \(s_n = 2s_{n-1}\) for \(n > 1 \) and \(s_1 = 1\).

c. \(s_n = s_{n-1} + s_{n-2}\) for \(n > 2\) and \(s_1 = s_2 = 1\).

d. \(s_n = \frac{s_{n-1}}{n}\) for \(n > 1 \) and \(s_1 = 1\).
**Example 6** Consider a recursive formula given by

\[ s_n = \begin{cases} \frac{s_{n-1}}{2} & \text{if } s_{n-1} \text{ is even} \\ 3s_{n-1} + 1 & \text{if } s_{n-1} \text{ is odd} \end{cases} \]

a. With \( s_1 = 4 \), write the first few terms of the sequence.

b. With \( s_1 = 5 \), write the first few terms of the sequence.

c. With \( s_1 = 7 \), write the first few terms of the sequence.

d. Try other initial terms. Does your answer always end up with \( 4, 2, 1, 4, 2, 1, \ldots \)?

We now consider the limit of a sequence.

**Definition** If \( s_n \) approaches a fixed number \( L \) as \( n \to \infty \), then we say that \( s_n \) converges to \( L \) and \( L \) is the limit of \( s_n \). When there is no such \( L \), we say that the sequence diverges.

**Remark** The concept of the limit of a sequence is not much different from that of a function as \( x \to \infty \) and the usual limit rules on functions still hold.

**Example 7** Do the following sequences converge or diverge? If a sequence converges, find its limit.

a. \( s_n = \frac{1}{n} \) 

d. \( s_n = \frac{(-1)^n}{n} \)

b. \( s_n = \frac{5n^2 - 100}{2n^2 - 3n + 5} \) 

e. \( s_n = \frac{\sin n}{n} \)

c. \( s_n = n^2 \) 

f. \( s_n = 1 + (-1)^n \)
Example 8 Examine the limit of each of the following sequences

a. \( s_n = \left(\frac{1}{4}\right)^n \)  

b. \( s_n = \left(-\frac{1}{2}\right)^n \)  

c. \( s_n = \left(\frac{4}{3}\right)^n \)  

d. \( s_n = (-2)^n \)

Remark In summary we get

\[
\lim_{n \to \infty} r^n = \\
\begin{cases} 
\infty & \text{if } r > 1 \\
1 & \text{if } r = 1 \\
0 & \text{if } -1 < r < 1 \\
\text{does not exist} & \text{if } r \leq -1 
\end{cases}
\]

Example 9 Examine the limit of each of the following sequences:

a. \( s_n = \frac{1+3^n}{2-5^n} \)  

b. \( s_n = \frac{2+4^n}{3^n} \)  

c. \( s_n = \frac{6^n+3^{2n}}{3^n+5^9^n} \)  

d. \( s_n = \frac{2^n+4^n}{3^n+2^n} \)

Definition Let \( s_n \) be a sequence.

a. \( s_n \) is said to be bounded from above if there is a constant \( M \) such that \( s_n \leq M \) for all \( n \).

b. \( s_n \) is said to be bounded from below if there is a constant \( K \) such that \( K \leq s_n \) for all \( n \).

c. \( s_n \) is said to be bounded if \( s_n \) is both bounded from above and from below.

d. \( s_n \) is said to be increasing (strictly increasing, respectively) if \( s_{n+1} \geq s_n \) (\( s_{n+1} > s_n \), respectively) for all \( n \).

e. \( s_n \) is said to be decreasing (strictly decreasing, respectively) if \( s_{n+1} \leq s_n \) (\( s_{n+1} < s_n \), respectively) for all \( n \).

f. \( s_n \) is called monotone if it is either increasing or decreasing.
Example 10 Which of the following sequences are bounded from above? bounded from below? increasing? decreasing?

a. 2, 4, 6, 8, 10, 12, . . .

b. 1, −1, 1, −1, 1, −1, . . .

c. \(\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \frac{5}{6}, -\frac{6}{7}, \ldots\)

d. 1, \(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots\)

e. \(\cos 1, \cos 2, \cos 3, \cos 4, \cos 5, \cos 6, \ldots\)

Theorem 1 A convergent sequence is bounded.

Remark The converse of Theorem 1 is not true: consider \(s_n = (-1)^n\).

Theorem 2 Let \(s_n\) be a sequence.

a. If \(s_n\) is increasing and bounded from above, then \(s_n\) converges.

b. If \(s_n\) is decreasing and bounded from below, then \(s_n\) converges.

Example 11 Let \(s_n = (1 + \frac{1}{n})^n\). It is known that \(s_n\) is increasing and \(s_n < 4\) for all \(n\). Therefore, \(\lim_{n \to \infty} s_n\) exists. This limit is called \(\ldots\).
Section 9.2 Geometric Series

Definition A finite series is the sum of finitely many terms of a sequence. For a given sequence \( a_n \), the sum of the first \( k \) terms, called the \( k \)th partial sum and denoted by \( S_k \), is defined by

\[
S_k = \sum_{i=1}^{k} a_i = a_1 + a_2 + a_3 + \cdots + a_{k-1} + a_k.
\]

Example 1 Let \( a_n = 2n + 1 \). Then

\[
\sum_{i=1}^{5} a_i = a_1 + a_2 + a_3 + a_4 + a_5 = 3 + 5 + 7 + 9 + 11 = 35.
\]

Remark

a. \( i \) is a dummy variable in the expression above, that is, we could have used \( \sum_{j=1}^{k} a_j \) or \( \sum_{n=1}^{k} a_n \) to express the same finite series.

b. The summation notation can be used in a different way. For example,

\[
\sum_{i=3}^{5} i^2 = 3^2 + 4^2 + 5^2 = 50 \quad \text{and} \quad \sum_{p \text{ prime}}^{p \leq 10} p = 2 + 3 + 5 + 7 = 17.
\]

c. \( S_n = S_{n-1} + a_n \), so \( a_n = S_n - S_{n-1} \).

Definition A geometric sequence or geometric progression (GP) is a sequence of the form

\[
a, ar, ar^2, ar^3, \ldots
\]

Here \( a \) is called the initial term and \( r \) is called the common ratio.

Example 2 Consider the sequence 4, 2, 1, \( \frac{1}{2}, \frac{1}{4}, \frac{1}{8} \), \ldots. Check that it is a GP. Find the initial term and the common ratio.

Remark

a. The general \( n \)th term of a GP with initial term \( a \) and common ratio \( r \) is given by \( a_n = ar^{n-1} \).

b. The sum of terms of a GP, \( a + ar + ar^2 + \cdots + ar^{n-1} \), can be written as \( \sum_{k=1}^{n} ar^{k-1} \).
Theorem 1 If $r \neq 1$, then $\sum_{k=1}^{n} a r^{k-1} = a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1} = \frac{a(1 - r^n)}{1 - r}$.

Proof

Remark If $r = 1$, then $\sum_{k=1}^{n} a r^{k-1} = a + a + \cdots + a = n a$.

Example 3 Compute each of the following sums:

a. $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243}$.

b. $1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512 + 1024$.

Definition Let $\{a_n\}$ be a sequence. An infinite series $\sum_{n=1}^{\infty} a_n$ is defined to be

$$\sum_{n=1}^{\infty} a_n = \lim_{k \to \infty} S_k,$$

where $S_k$ denotes the partial sum $S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_{k-1} + a_k$. In other words,

$$\sum_{n=1}^{\infty} a_n = \lim_{k \to \infty} S_k = \lim_{k \to \infty} \sum_{n=1}^{k} a_n.$$

The Geometric Series Test

Let $a_n$ be a GP with initial term $a$ and common ration $r$. Then

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a r^{n-1} = \begin{cases} \frac{a}{1 - r}, & \text{if } -1 < r < 1 \\ \text{diverges}, & \text{otherwise} \end{cases}$$

Proof
Example 4 Compute each of the following infinite geometric series, if it converges.

a. $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \frac{1}{81} - \frac{1}{243} + \cdots$

b. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$

c. $1 - 2 + 4 - 8 + 16 - 32 + \cdots$

Example 5 In the following picture, find the infinite sum $\sum_{n=1}^{\infty} \frac{P_n P_{n+1}}{P_{n+1} P_{n+2}}$. Here $\angle P_n P_{n+1} P_{n+2}$ is the right angle for all $n$. 
Section 9.3 Convergence of Series

Recall the following definition:

**Definition** An infinite series \( \sum_{n=1}^{\infty} a_n \) is defined to be

\[
\sum_{n=1}^{\infty} a_n = \lim_{k \to \infty} \sum_{n=1}^{k} a_n.
\]

The \( k \)th partial sum \( S_k \) is defined to be \( S_k = \sum_{n=1}^{k} a_n \). Therefore,

\[
\sum_{n=1}^{\infty} a_n = \lim_{k \to \infty} S_k.
\]

In this section and the next, we are mostly interested in determining whether a given infinite series is convergent or divergent. We will develop several tests. Before we get started, note that changing finitely many terms in a series does not change whether or not it converges, although it may change the value of its sum if it does converge.

**Example 1** Recall that

\[
1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots = 2.
\]

If you change it to

\[
-1 + 3 + \frac{1}{4} + 2 + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots,
\]

then still the new infinite sum converges, although the value of the sum is now different.

**Theorem 1** If \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) both converge and if \( c \) is a constant, then

a. \( \sum_{n=1}^{\infty} (a_n \pm b_n) \) converges to \( \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n \).

b. \( \sum_{n=1}^{\infty} ca_n \) converges to \( c \sum_{n=1}^{\infty} a_n \).

**Proof**
Example 2 Let $c$ be a nonzero constant. If $\sum_{n=1}^{\infty} a_n$ diverges, then does $\sum_{n=1}^{\infty} ca_n$ diverge as well?

The $n^{th}$ Term Test

a. If $\lim_{n \to \infty} |a_n| \neq 0$ or $\lim_{n \to \infty} |a_n|$ does not exist, then $\sum_{n=1}^{\infty} a_n$ diverges.

b. If $\lim_{n \to \infty} |a_n| = 0$, then the test fails.

Proof

Example 3 Does the series $\sum_{n=1}^{\infty} \frac{n+2}{n}$ converge? How about $\sum_{n=1}^{\infty} \frac{1}{n}$? Or $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$?

The Integral Test

Suppose $a_n = f(n)$, where $f(x)$ is decreasing and positive for $x \geq 1$.

a. If $\int_{1}^{\infty} f(x) \, dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges, too.

b. If $\int_{1}^{\infty} f(x) \, dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges, too.

Proof
**Example 4** Does the series \( \sum_{n=1}^{\infty} \frac{1}{e^n} \) converge?

**Example 5** Does the series \( \sum_{n=2}^{\infty} \frac{1}{n \ln n} \) converge?

**Example 6** For what values of \( p \) does the series \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) converge?

We can summarize **Example 6** as follows:

<table>
<thead>
<tr>
<th>The ( p )-series Test</th>
</tr>
</thead>
<tbody>
<tr>
<td>The ( p )-series ( \sum_{n=1}^{\infty} \frac{1}{n^p} ) converges if ( p &gt; 1 ) and diverges if ( p \leq 1 ).</td>
</tr>
</tbody>
</table>
Section 9.4 Tests for Convergence

In this section, we will develop more tests for convergence of a series.

Comparison Test

Suppose $0 \leq a_n \leq b_n$ for all $n$.

a. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges, too.

b. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges, too.

Proof

Example 1 Decide whether the following series converge:

a. $\sum_{n=1}^{\infty} \frac{2n - 1}{n^3 + 3}$

b. $\sum_{n=1}^{\infty} \frac{6n^2 + 1}{2n^3 - 1}$

c. $\sum_{n=1}^{\infty} \frac{1}{n^2 + n}$
Example 2 Determine whether $\sum_{n=1}^{\infty} \frac{2n + 10}{5n^3 - 3}$ converges. Can we apply comparison test?

Limit Comparison Test

Suppose $a_n > 0$ and $b_n > 0$ for all $n$.

a. If $\lim_{n \to \infty} \frac{a_n}{b_n} = c$ with $c > 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge.

b. Suppose $\lim_{n \to \infty} \frac{a_n}{b_n} = 0$.

i. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges, too.

ii. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges, too.

c. Suppose $\lim_{n \to \infty} \frac{a_n}{b_n} = \infty$.

i. If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges, too.

ii. If $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges, too.

Proof

Example 3 Complete Example 2.
Example 4 Decide whether the following series converge:

a. \[ \sum_{n=1}^{\infty} \frac{n^2 - 5}{n^3 + n + 4} \]

b. \[ \sum_{n=1}^{\infty} \sin \left( \frac{1}{n} \right) \]

c. \[ \sum_{n=1}^{\infty} \frac{\ln n}{n^2} \]

d. \[ \sum_{n=2}^{\infty} \frac{1}{\ln n} \]

We introduce several more tests. Note that the next three tests can be applied to series with both positive and negative terms.

The Ratio Test

Suppose that the limit of \( \left| \frac{a_{n+1}}{a_n} \right| \) exists, say \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L \).

a. If \( L < 1 \), then \( \sum_{n=1}^{\infty} a_n \) converges.

b. If \( L > 1 \) (including \( L = \infty \)), then \( \sum_{n=1}^{\infty} a_n \) diverges.

c. If \( L = 1 \), then the test fails.

Proof
Example 5 Decide whether the following series converge:

a. \( \sum_{n=1}^{\infty} \frac{n^2}{2^n} \)

b. \( \sum_{n=1}^{\infty} \frac{(-1)^n n^n}{n!} \)

The Absolute Convergence Test

If \( \sum_{n=1}^{\infty} |a_n| \) converges, then so does \( \sum_{n=1}^{\infty} a_n \).

Proof

Example 6 Since \( \sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges, \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \) converges, too.
Example 7 Decide whether the following series converge:

a. \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/2}} \]

b. \[ \sum_{n=1}^{\infty} \frac{\sin n}{n^2} \]

The Alternating Series Test

If \( a_n \) is a positive decreasing sequence with \( \lim_{n \to \infty} a_n = 0 \), then \( \sum_{n=1}^{\infty} (-1)^n a_n \) converges.

Proof

Example 8 Decide whether \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \) converges.

Example 9 Show that \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \) converges.
Remark For any infinite series $\sum_{n=1}^{\infty} a_n$, exactly one of the following is true,

a. $\sum_{n=1}^{\infty} |a_n|$ converges (so $\sum_{n=1}^{\infty} a_n$ also converges by the Absolute Convergence Test),

b. $\sum_{n=1}^{\infty} |a_n|$ diverges but $\sum_{n=1}^{\infty} a_n$ still converges, or

c. $\sum_{n=1}^{\infty} a_n$ diverges.

Definition

a. If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ is said to converge absolutely.

b. If $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges, then $\sum_{n=1}^{\infty} a_n$ is said to converge conditionally.

Example 10 $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges conditionally, while $\sum_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n}}$ converges absolutely.

Example 11 Determine whether $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln n}$ converge absolutely, conditionally, or diverges.

Remark There are still many convergence tests not discussed here, such as Root test, Kummer’s test, Gauss’s test, Raabe’s test, and so on.
Summary

The Geometric Series Test

Let \( a_n \) be a GP with initial term \( a \) and common ration \( r \). Then

\[
\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a r^{n-1} = \begin{cases} \frac{a}{1-r}, & \text{if } -1 < r < 1 \\ \text{diverges}, & \text{otherwise} \end{cases}
\]

The \( n^{th} \) Term Test

a. If \( \lim_{n \to \infty} |a_n| \neq 0 \) or \( \lim_{n \to \infty} |a_n| \) does not exist, then \( \sum_{n=1}^{\infty} a_n \) diverges.

b. If \( \lim_{n \to \infty} |a_n| = 0 \), then the test fails.

The Integral Test

Suppose \( a_n = f(n) \), where \( f(x) \) is decreasing and positive for \( x \geq 1 \).

a. If \( \int_{1}^{\infty} f(x) \, dx \) converges, then \( \sum_{n=1}^{\infty} a_n \) converges, too.

b. If \( \int_{1}^{\infty} f(x) \, dx \) diverges, then \( \sum_{n=1}^{\infty} a_n \) diverges, too.

The \( p \)-series Test

The \( p \)-series \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) converges if \( p > 1 \) and diverges if \( p \leq 1 \).
Comparison Test

Suppose \( 0 \leq a_n \leq b_n \) for all \( n \).

a. If \( \sum_{n=1}^{\infty} b_n \) converges, then \( \sum_{n=1}^{\infty} a_n \) converges, too.

b. If \( \sum_{n=1}^{\infty} a_n \) diverges, then \( \sum_{n=1}^{\infty} b_n \) diverges, too.

Limit Comparison Test

Suppose \( a_n > 0 \) and \( b_n > 0 \) for all \( n \).

a. If \( \lim_{n \to \infty} \frac{a_n}{b_n} = c \) with \( c > 0 \), then \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) either both converge or both diverge.

b. Suppose \( \lim_{n \to \infty} \frac{a_n}{b_n} = 0 \).

i. If \( \sum_{n=1}^{\infty} b_n \) converges, then \( \sum_{n=1}^{\infty} a_n \) converges, too.

ii. If \( \sum_{n=1}^{\infty} a_n \) diverges, then \( \sum_{n=1}^{\infty} b_n \) diverges, too.

c. Suppose \( \lim_{n \to \infty} \frac{a_n}{b_n} = \infty \).

i. If \( \sum_{n=1}^{\infty} a_n \) converges, then \( \sum_{n=1}^{\infty} b_n \) converges, too.

ii. If \( \sum_{n=1}^{\infty} b_n \) diverges, then \( \sum_{n=1}^{\infty} a_n \) diverges, too.
The Ratio Test

Suppose that the limit of $\frac{|a_{n+1}|}{|a_n|}$ exists, say $\lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = L$.

a. If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

b. If $L > 1$ (including $L = \infty$), then $\sum_{n=1}^{\infty} a_n$ diverges.

c. If $L = 1$, then the test fails.

The Absolute Convergence Test

If $\sum_{n=1}^{\infty} |a_n|$ converges, then so does $\sum_{n=1}^{\infty} a_n$.

The Alternating Series Test

If $a_n$ is a positive decreasing sequence with $\lim_{n \to \infty} a_n = 0$, then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.
Section 9.5 Power Series and Interval of Convergence

**Definition** A power series about \( x = a \) is a series of the form

\[
\sum_{n=0}^{\infty} C_n (x - a)^n = C_0 + C_1 (x - a) + C_2 (x - a)^2 + \cdots + C_n (x - a)^n + \cdots ,
\]

where \( C_n \) are constants. In this case, \( a \) is called the center of the power series.

**Example 1** \(2 + (x - 1) + \left(\frac{x - 1}{2}\right)^2 + \left(\frac{x - 1}{3}\right)^3 + \left(\frac{x - 1}{4}\right)^4 + \cdots\) is a power series about \( x = 1 \). Here \( C_0 = 2 \) and \( C_n = \frac{1}{n} \) for \( n \geq 1 \).

**Example 2** \(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots\) is a power series with the center 0. Here \( C_n = \frac{1}{n!} \) for all \( n \).

**Remark** A power series is an infinite series which can be regarded as a polynomial of infinite degree. Since a power series is an infinite series with a variable \( x \) in it, its convergence depends on the value of \( x \). Our first interest is in determining the values of \( x \) for which a given power series converges.

**Example 3** Determine whether the power series \( \sum_{n=0}^{\infty} \frac{x^n}{2^n} \) converges or diverges for

a. \( x = -1 \)

b. \( x = 3 \)

**Example 4** For which values of \( x \) does the power series \( \sum_{n=0}^{\infty} \frac{x^n}{2^n} \) converge?
Example 5 For which values of $x$ does the power series $\sum_{n=0}^{\infty} n! (x - 3)^n$ converge?

Example 6 For which values of $x$ does the power series $\sum_{n=0}^{\infty} \frac{(x + 2)^n}{n!}$ converge?

We observed that a power series about $(x - a)$ converges for at least one $x$, say $x = a$. In fact, the following is known:

Theorem 1 Let a power series $\sum_{n=0}^{\infty} C_n (x - a)^n$ be given. Then exactly one of the following is true for the power series:

a. There is a positive number $R$, called the radius of convergence, such that the series converges for $|x - a| < R$ and diverges for $|x - a| > R$.

b. The series converges only for $x = a$. In this case we say that the radius of convergence $R$ equals 0.

c. The series converges for all values of $x$. In this case we say that the radius of convergence $R$ equals $\infty$.

Example 7 Determine the radius of convergence for the power series in Example 4, 5, 6 above.
There is an easy method for computing the radius of convergence:

**Theorem 2** Let a power series $\sum_{n=0}^{\infty} C_n(x - a)^n$ be given. Let $a_n = C_n(x - a)^n$.

a. If $\lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \infty$, then $R = 0$.

b. If $\lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = 0$, then $R = \infty$.

c. If $\lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = K|x - a|$, then $R = \frac{1}{K}$.

**Proof**

**Example 8** Show that the following power series converges for all $x$:

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots$$

**Example 9** Determine the radius of convergence of the series

$$(x - 1) - \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{3} - \frac{(x - 1)^4}{4} + \cdots + (-1)^{n+1} \frac{(x - 1)^n}{n} + \cdots$$
Suppose that a power series $\sum_{n=0}^{\infty} C_n(x-a)^n$ has a finite radius of convergence $R$. That is to say, we know that the power series converges for $|x-a| < R$ and diverges for $|x-a| > R$. In other words, the power series converges for $a - R < x < a + R$ and diverges for $x < a - R$ or $x > a + R$. What happens at the boundary points, say $x = a - R$ and $x = a + R$? There is no simple theorem that answers this question. To investigate the behavior of the power series at these boundary points, one needs to plug the boundary points into the power series.

**Example 10** We know that the radius of convergence of

\[
(x - 1) - \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{3} - \frac{(x - 1)^4}{4} + \cdots + (-1)^{n+1} \frac{(x - 1)^n}{n} + \cdots
\]

equals 1. This tells us that the power series converges for $0 < x < 2$ and diverges for $x < 0$ or $x > 2$. What happens if $x = 0$ or $x = 2$?

**Definition** The interval of convergence of a power series is the collection of all $x$ values for which the given power series converges.

**Example 11** Let a power series $\sum_{n=0}^{\infty} C_n(x-a)^n$ be given. If the radius of convergence $R = \infty$, then the interval of convergence is $(-\infty, \infty)$. If $R = 0$, then the interval of convergence is a single point $\{a\}$. If the radius of convergence $R$ is positive number, then the interval of convergence is one of the following intervals:

\[
(a - R, a + R), \quad [a - R, a + R), \quad (a - R, a + R], \quad \text{or} \quad [a - R, a + R].
\]

**Example 12** The interval of convergence of the power series in **Example 10** is $(0, 2]$.