Unmixed Edge Ideals

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Definition

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Example
Definition

If $G$ is a graph on $n$ vertices then the *complement* of $G$, denoted $G^c$, is the graph on $n$ vertices where two vertices are connected in $G^c$ if and only if they are *not* connected in $G$. 
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Example

A graph and its complement:
Definition

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Example

\( K_3 \) and \( K_5 \):

![Diagram of \( K_3 \) and \( K_5 \)]
Definition
Let $R$ be a commutative ring. Then a subset $I \subseteq R$ is an ideal of $R$ if it satisfies the following two properties:

- $I$ is an abelian group under addition.
- If $r \in R$ and $s \in I$ then $rs$ is in $I$.

Example
If $R = \mathbb{Z}$ then the multiples of 6 form an ideal.
Definition

We say an ideal $I$ of $R$ is *generated* by elements $s_1, \ldots, s_m$ if every element $s \in I$ can be expressed as

$$s = r_1 s_1 + \ldots + r_m s_m$$

for some $r_1, \ldots, r_m$ in $R$. If $I$ is generated by $s_1, \ldots, s_m$ then we write $I = \langle s_1, \ldots, s_m \rangle$. 

Ideals (Cont’d)

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for some \( r_1, \ldots, r_m \) in \( R \). If \( I \) is generated by \( s_1, \ldots, s_m \) then we write \( I = \langle s_1, \ldots, s_m \rangle \).

Example
If \( R = \mathbb{R}[x, y] \) then

\[
I = \{ f(x, y)x^2 + g(x, y)(x + y) \mid f(x, y), g(x, y) \in \mathbb{R}[x, y] \}
\]
is an ideal. In this case \( I = \langle x^2, x + y \rangle \).
Definition

Let $G$ be a graph on vertices labeled $x_1$ through $x_n$. The edge ideal of $G$, denoted $I(G)$, is the ideal of $\mathbb{R}[x_1, \ldots, x_n]$ whose generators are given as follows: $x_ix_j$ is a generator of $I(G)$ if and only if $x_i$ is connected to $x_j$ in $G$. 

Example

The graph $G$ below:

\[x_1\quad x_2\quad x_3\quad x_4\quad x_5\]

has edge ideal:

\[I(G) = \langle x_1x_2, x_1x_3, x_2x_3, x_3x_4, x_4x_5 \rangle\]
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**Example**

The graph $G$ below:

```
 x1 -- x2 -- x3 -- x4
     /   \
    /     \
 x5     x4
```

has edge ideal:

$$I(G) = \langle x_1x_2, x_1x_3, x_2x_3, x_3x_4, x_4x_5 \rangle$$
Definition

Given an edge ideal $I(G)$ in $\mathbb{R}[x_1, \ldots, x_n]$ we define the edge variety of $I(G)$, denoted $V(G)$, to be the common solution set in $\mathbb{R}^n$ of the generators of $I(G)$. 

Example

Suppose $G$ is the graph displayed below:

Then $I(G) = \langle xy, yz \rangle \subseteq \mathbb{R}[x, y, z]$ and $V(G)$ is the union of the $xz$-plane and the $y$-axis.
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Suppose $G$ is the graph displayed below:

```
     y
    /|
   / | \
  /  |  \
 x---z
```

Then $I(G) = \langle xy, yz \rangle \subseteq \mathbb{R}[x, y, z]$ and $V(G)$ is the union of the $xz$-plane and the $y$-axis.
Fact

If $G$ is a graph on $n$ vertices then its edge variety $V(G)$ is the finite union of coordinate subspaces of $\mathbb{R}^n$. 
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If $I(G)$ is an edge ideal then $I(G)$ can be uniquely expressed as the intersection of ideals generated by variables.

Definition

We refer to this intersection as the *primary decomposition* of $I(G)$, and we refer to the individual ideals in the intersection as the *associated prime ideals* of $I(G)$.
Example

Suppose $G$ is the graph displayed below:

```
    y
   /|
  x  z
```

Then

$$I(G) = \langle xy, yz \rangle$$

and the primary decomposition of $I(G)$ is

$$\langle x, z \rangle \cap \langle y \rangle.$$
Example

Suppose \( G \) is the graph displayed below:

\[
\begin{align*}
\text{Then } & \quad I(G) = \langle x_1x_2, x_1x_3, x_2x_3, x_3x_4, x_4x_5 \rangle \\
\text{and the primary decomposition of } I(G) \text{ is } & \quad \langle x_1, x_2, x_4 \rangle \cap \langle x_1, x_3, x_4 \rangle \cap \langle x_2, x_3, x_5 \rangle \cap \langle x_2, x_3, x_4 \rangle \cap \langle x_2, x_3, x_5 \rangle.
\end{align*}
\]
We say an edge ideal $I(G)$ in $\mathbb{R}[x_1, \ldots, x_n]$ is *unmixed* if all of its associated prime ideals have the same number of variables. We say $I(G)$ is *unmixed of dimension r* if each of its associated prime ideals is generated by $n - r$. 
Unmixed Edge Ideals

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Example

We saw in the previous two slides that the graph on the left is not unmixed, and that the graph on the right is unmixed of dimension 2.
Question

Given a graph $G$, how do you tell if $I(G)$ is unmixed of dimension $r$?
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**Remark**
If $G$ is a graph then the edge ideal $I(G)$ being unmixed of dimension $r$ is equivalent to each of the (maximal) coordinate subspaces of $V(G)$ having dimension $r$. 
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**Remark**
If $G$ is a graph then the edge ideal $I(G)$ being unmixed of dimension $r$ is equivalent to each of the (maximal) coordinate subspaces of $V(G)$ having dimension $r$.

**Strategy**
Given an arbitrary edge ideal $I(G)$, try to state in terms of the generators of $I(G)$ just what $V(G)$ looks like. In particular, try to express what the maximal coordinate subspaces of $V(G)$ look like.
An Answer for Dimension 2

Theorem

Let $G$ be a graph. Then, $I(G)$ is unmixed of dimension 2 if and only if the following two conditions hold:

- $G^c$ contains no triangles.
- $G^c$ contains no isolated vertices.

Example

The complement of $G$ contains no triangle and no isolated vertices. We saw earlier that $G$ is unmixed of dimension 2, as predicted.
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Remark

We could rephrase the conditions of the previous theorem as follows:

- $G^c$ contains no copies of $K_3$.
- Whenever $G^c$ contains a copy of $K_1$, it lies inside some copy of $K_2$. 

(Here $i$ is any positive integer less than $r$.)
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Theorem

Let $G$ be a graph. Then, $I(G)$ is unmixed of dimension $r$ if and only if the following two conditions hold:

- $G^c$ contains no copies of $K_{r+1}$
- Whenever $G^c$ contains a copy of $K_{r-i}$, it lies inside some copy of $K_r$.

(Here $i$ is any positive integer less than $r$.)
Example

The graph on the left is unmixed of dimension 3. The graph on the right is not unmixed.
Applications

- Describing the Alexander dual of $I(G)$
- Determining when $I(G)$ is Cohen-Macaulay
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