1. Introduction

Throughout this paper, let $S = \mathbb{R}[x_0, \ldots, x_n]$ be the polynomial ring in $n+1$ variables whose coefficients are real numbers. In order to define a division algorithm on $S$ for $n \geq 1$, we must have a way of ordering the monomials in $S$.

**Definition 1.** A monomial ordering $>_{\text{on}}$ on $S$ is a total ordering of the monomials in $S$ that satisfies

(1) every monomial of positive degree $>_{\text{on}}$ every constant in $\mathbb{R}$

(2) if $m$ and $n$ are monomials such that $m > n$, then $pm > pn$ for all monomials $p \in S$.

To simplify our notation, if $m = x_0^{\alpha_0}x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ is a monomial in $S$, then we write $m \equiv x^{(\alpha_0, \alpha_1, \ldots, \alpha_n)}$.

**Example 2** (Lexicographic Term Ordering). Using the above notation, let $x^\alpha$ and $x^\beta$ be monomials in $S$ where $\alpha, \beta \in \mathbb{Z}_{\geq 0}^{n+1}$. Then we say $x^\alpha >_{\text{lex}} x^\beta$ if and only if the left-most nonzero entry of $\alpha - \beta$ is positive.

**Example 3** (Reverse Lexicographic Term Ordering). Let $x^\alpha$ and $x^\beta$ be as in Example 2. We say that $x^\alpha >_{\text{revlex}} x^\beta$ if and only if the right-most nonzero entry of $\alpha - \beta$ is negative.

A monomial ordering allows us to define the initial term for any polynomial $f \in S$.

**Definition 4.** Fix a monomial ordering $>_{\text{on}}$ on $S$. Let $f = c_0m_0 + c_1m_1 + \cdots + c_km_k$ be a polynomial in $S$ where $c_0, \ldots, c_k \in \mathbb{R}$ and $m_0, \ldots, m_k$ are monomials in $S$. The the initial term of $f$, denoted in $f$, is the monomial $m_i$ such that $m_i > m_j$ for all $j \neq i$. 

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Consider an ideal \( I \subseteq S \) that is generated by a finite number of polynomials \( f_0, \ldots, f_k \in S \). That is, suppose
\[
I = \langle f_0, f_1, \ldots, f_k \rangle = \{ p_0 f_0 + p_1 f_1 + \cdots + p_k f_k | p_0, \ldots, p_k \in S \}.
\]
We can then define the initial ideal of \( I \), denoted \( \text{in} \ I \), to be
\[
\text{in} \ I = \langle \text{in} \ f | f \in I \rangle.
\]
For \( I = \langle f_0, \ldots, f_k \rangle \), it is easy to see that
\[
\langle \text{in} \ f_0, \ldots, \text{in} \ f_k \rangle \subseteq \text{in} \ I,
\]
but equality need not hold. It is then natural to ask when equality does hold.

**Definition 5.** Fix a monomial ordering \( > \) on \( S \) and let \( I \) be an ideal of \( S \). Consider \{\( g_0, \ldots, g_r \)\} such that \( I = \langle g_0, \ldots, g_r \rangle \). We say \{\( g_0, \ldots, g_r \)\} is a Gröbner basis for \( I \) if
\[
\text{in} \ I = \langle \text{in} \ g_0, \ldots, \text{in} \ g_r \rangle.
\]

Efficient algorithms exist for finding a Gröbner basis \{\( g_0, \ldots, g_r \)\} for \( I = \langle f_0, \ldots, f_k \rangle \).

We can learn things about ideals by studying their free resolutions.

**Definition 6.** Let \( I = \langle f_1, \ldots, f_r \rangle \) be an ideal of \( S \). A free resolution of \( I \) is a sequence of maps \( \phi_i \)
\[
\begin{array}{cccc}
S & S & \cdots \\
\oplus & \oplus & \cdots \\
\phi_2 & \phi_1 & \cdots \\
\oplus & \oplus & \cdots \\
S & S & S
\end{array}
\]
such that image \( \phi_{i+1} = \ker \phi_i \).

**Definition 7.** An element of \( \ker \phi_i \) for some \( i \) is called a syzygy. We say \( \tau \) is an \( i \)th syzygy if \( \tau \in \ker \phi_i \).

Note that the set of all \( i \)th syzygies forms an \( S \)-module.

Information about free resolutions and syzygies is frequently encoded in a betti diagram:
\[
\begin{array}{cccccc}
0 & 1 & \cdots & s \\
\beta_{0,0} & \beta_{1,1} & \cdots & \beta_{s,s} \\
\beta_{0,1} & \beta_{1,2} & \cdots & \beta_{s,s+1} \\
\vdots & \vdots & \ddots & \vdots \\
\beta_{0,j} & \beta_{1,j+1} & \cdots & \beta_{s,j+s}
\end{array}
\]
Here $\beta_{i,k}$ is the number of 1-dimensional syzygy modules at place $i$ in the free resolution that have degree $k$.

We also include an important result about syzygies that minimally generate their respective syzygy module.

**Theorem 8** ([2], Theorem 15.10 (Schreyer)). Let $g_1, \ldots, g_t$ be a Gröbner basis. Let $>$ be the monomial order on $\bigoplus_{j=1}^t F$ defined by taking $m \varepsilon_u > n \varepsilon_v$ if and only if

$$\text{in}(mg_u) > \text{in}(ng_v)$$

with respect to the given order on $F$

or

$$\text{in}(mg_u) = \text{in}(ng_v) \text{ (up to a scalar)} \text{ but } u < v.$$

If

$$\tau_{ij} = m_{ji} \varepsilon_i - m_{ij} \varepsilon_j - \sum_u f_{ij}^{(ij)} \varepsilon_u$$

where $i < j$ such that $\text{in}(g_i)$ and $\text{in}(g_j)$ involve the same basis element of $F$, then the $\tau_{ij}$ generate the syzygies on the $g_i$. In fact, the $\tau_{ij}$ are a Gröbner basis for the syzygies with respect to the order $>$, and $\text{in}(\tau_{ij}) = m_{ji} \varepsilon_i$.

2. **Generic Initial Ideals**

I studied the ideal $I$ generated by a smooth genus three curve of degree four that is embedded in $\mathbb{P}^9$. Initially, I was interested in this ideal and its secants and was hoping to see patterns similar to those of the genus two curve of degree $g$ in $\mathbb{P}^{10}$.

To study $I$, I tried to find a square-free initial ideal so that I could think of the initial ideal as the edge ideal of a graph. Unfortunately, the ideal proved too large to be able to compute the Gröbner fan directly, and running gfan interactively and “walking through walls” proved to be inefficient and seemed unlikely to lead to the discovery of a square-free initial ideal.

I have since been using the generic initial ideal of $I$.

**Definition 9.** Fix a term ordering $>$ on $S$ and let $I$ be a homogeneous ideal. Then there exists an open dense subset $B$ of $\text{GL}_n(k)$ such that for a change-of-coordinate matrix $M \in B$

$$\text{gin} I = \text{in}_>(MI)$$

for all $M \in B$.

We say $\text{gin} I$ is the **generic initial ideal** of $I$. 
When the gin of an ideal is taken with respect to the reverse lexicographic term ordering, generic initial ideals preserve some important properties of the original ideal. More specifically, we have the following theorem due to Bayer and Stillman [1, 2]:

**Theorem 10.** The betti diagram associated to $\text{gin}_{\text{revlex}} I$ has exactly the same number of rows and columns as the betti diagram associated to $I$.

$I$, $\text{gin} I$, and their first secants give the following betti diagrams:

betti diagram of $I$:

\[
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
0 & 1 & . & . & . & . & . & . & . \\
1 & . & 33 & 144 & 294 & 336 & 210 & 48 & . \\
2 & . & . & . & . & . & 21 & 16 & 3 \\
\end{array}
\]

betti diagram of $\text{gin}_{\text{revlex}} I$:

\[
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
0 & 1 & . & . & . & . & . & . & . \\
1 & . & 33 & 147 & 315 & 399 & 315 & 153 & 42 & 5 \\
2 & . & 3 & 21 & 63 & 105 & 105 & 63 & 21 & 3 \\
\end{array}
\]

betti diagram of $I^{(2)}$

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 \\
0 & 1 & . & . & . & . & . \\
1 & . & . & . & . & . & . \\
2 & . & 38 & 108 & 102 & 10 & . \\
3 & . & . & . & 30 & . & . \\
4 & . & . & . & 3 & 18 & 6 \\
\end{array}
\]

betti diagram of $\text{gin}_{\text{revlex}} (I^{(2)})$

\[
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 \\
0 & 1 & . & . & . & . & . \\
1 & . & . & . & . & . & . \\
2 & . & 38 & 120 & 156 & 100 & 30 & 3 \\
3 & . & 12 & 60 & 120 & 120 & 60 & 12 \\
4 & . & 6 & 30 & 60 & 60 & 30 & 6 \\
\end{array}
\]

betti diagram of $(\text{gin}_{\text{revlex}} I)^{(2)}$
Here, note that $\text{gin}_{\text{relex}}(I^{(2)}) \neq (\text{gin} I)^{(2)}$.

Future research could be done to further understand $I$, such as computing more secant ideals and comparing the resulting betti diagrams for the original ideal or, if that proves computationally difficult, of the generic initial ideals. Also, although the above clearly shows that $\text{gin}(I^{(2)}) \neq (\text{gin} I)^{(2)}$ and that the associated betti diagrams of these two ideals have a different number of columns, the two betti diagrams do have the same number of rows, so it is natural to ask if this is a coincidence or something that holds for any ideal (or for a certain class of ideals).

3. **Secants of Chordal Graphs**

Since the edge splitting techniques presented in [3] can be used recursively on edge ideals of chordal graphs, I have been studying certain kinds of chordal graphs given below.

**Example 11.** Chordal Graph $G_3$:  

![Chordal Graph G3](image)

Chordal Graph $G_4$:  

![Chordal Graph G4](image)
I used the Macaulay 2 software to compute the betti diagrams associated to the edge ideals and their secants for $n = 3, \ldots, 19$. The betti diagrams for $n = 10$ and $n = 11$ are given as examples, and then we present some theorems and conjectures about the betti diagram of $I_{n}^{(2)}$.

**Example 12.** betti diagram for $I_{10}$:

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>total: 1</td>
<td>19</td>
<td>102</td>
<td>275</td>
<td>417</td>
<td>371</td>
<td>194</td>
<td>56</td>
<td>7</td>
</tr>
<tr>
<td>0: 1</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>1: .</td>
<td>19</td>
<td>41</td>
<td>30</td>
<td>7</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>2: .</td>
<td>.</td>
<td>61</td>
<td>238</td>
<td>378</td>
<td>313</td>
<td>142</td>
<td>33</td>
<td>3</td>
</tr>
<tr>
<td>3: .</td>
<td>.</td>
<td>.</td>
<td>7</td>
<td>32</td>
<td>58</td>
<td>52</td>
<td>23</td>
<td>4</td>
</tr>
</tbody>
</table>

betti diagram for $I_{10}^{(2)}$:

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>total: 1</td>
<td>9</td>
<td>23</td>
<td>26</td>
<td>12</td>
<td>1</td>
</tr>
<tr>
<td>0: 1</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>1: .</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>2: .</td>
<td>9</td>
<td>8</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>3: .</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>4: .</td>
<td>.</td>
<td>15</td>
<td>25</td>
<td>10</td>
<td>.</td>
</tr>
<tr>
<td>5: .</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>6: .</td>
<td>.</td>
<td>.</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

**Example 13.** betti diagram for $I_{11}$:

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>total: 1</td>
<td>21</td>
<td>131</td>
<td>403</td>
<td>705</td>
<td>759</td>
<td>524</td>
<td>231</td>
<td>60</td>
<td>7</td>
</tr>
</tbody>
</table>
Theorem 14. Let \( I_n \) be the edge ideal of the chordal graph \( G_n \) as described above, and let \( I_n^{(2)} \) denote the secant ideal of \( I_n \). Then for the betti diagram of the free resolution of \( I_n^{(2)} \), we have

\[
\beta_{1,3} = n - 1
\]

when \( n \geq 3 \), and

\[
\beta_{2,4} = n - 2
\]

when \( n \geq 4 \).

Proof. For the edge ideal \( I_n \) of a chordal graph \( G_n \), we know that the second secant ideal is given by the induced subgraphs of \( G_n \) that are not three-colorable [4]. Thus, for the chordal graphs above, \( I_n^{(2)} \) is generated by the degree-three monomials corresponding to the three cycles on the vertices of \( G_n \). Then clearly \( I_n^{(2)} \) has \( n - 1 \) generators. These generators also generate \( F_1 \), the first module of syzygies for the free resolution of \( I_n^{(2)} \). Thus,

\[
\beta_{1,3} = n - 1.
\]

Let \( I_n^{(2)} = \langle f_1, \ldots, f_t \rangle \) By Theorem 8 if \( M = \langle f_1, \ldots, f_t \rangle \) is a submodule of \( S^n \) and \( g_1, \ldots, g_r \) is a Gröbner basis for \( M \) then the S-polynomials of \( g_1, \ldots, g_r \),

\[
S(g_i, g_j) = \sum a^{(ij)}_k f_k
\]

each correspond to a syzygy, and, further, these syzygies generate all syzygies on \( \langle g_1, \ldots, g_r \rangle \). Since secant ideals of edge ideals are always monomial ideals, we know \( \{ f_1, \ldots, f_t \} \) is a Gröbner basis for \( I_n^{(2)} \). Thus, we can find a generating set of syzygies for \( F_2 \).
Let $S(g_i, g_k)$ be the syzygy corresponding to the the $S$-polynomial $S(g_i, g_k)$. Consider the monomial pairs $x_i x_{i+1} x_{i+2}$ and $x_{i+1} x_{i+2} x_{i+3}$ for some $i \in \{1, \ldots, n-3\}$. Then

$$S(x_i x_{i+1} x_{i+2}, x_{i+1} x_{i+2} x_{i+3}) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -x_{i+3} \\ x_i \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where $-x_{i+3}$ is the $i$th component of the syzygy. For all $i \in \{1, \ldots, n-3\}$, this gives the set

$$\begin{pmatrix} -x_3 & 0 & 0 & 0 \\ x_0 & -x_4 & 0 & 0 \\ 0 & x_1 & 0 \\ 0 & 0 & \vdots & \vdots \\ 0 & 0 & 0 & -x_n \\ 0 & 0 & \vdots & x_{n-3} \end{pmatrix}.$$  

Setting

$$\begin{pmatrix} -x_3 \\ x_0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} -x_4 \\ x_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \lambda_{n-3} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ -x_n \\ x_{n-3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

for $\lambda_0, \ldots, \lambda_{n-3} \in \mathbb{C}$, we see that

$$-\lambda_0 x_3 = 0$$

$$\lambda_0 x_0 - \lambda_1 x_4 = 0$$

$$\vdots$$

$$\lambda_{n-4} x_{n-4} - \lambda_{n-3} x_n = 0$$

$$\lambda_{n-3} x_{n-3} = 0.$$
Thus,

\[- \lambda_0 x_3 = 0 \Rightarrow \lambda_0 = 0\]

\[\lambda_0 x_0 - \lambda_1 x_4 = -\lambda_1 x_4 = 0 \Rightarrow \lambda_1 = 0\]

\[\vdots\]

\[\lambda_{n-4} x_{n-4} - \lambda_{n-3} x_n = -\lambda_{n-3} = 0 \Rightarrow \lambda_{n-3} = 0,\]

so the set of \(n-2\) linear syzygies is linearly independent. Thus for \(n \geq 4\),

\[\beta_{2,2} = n - 2.\]

Now consider the monomials of two three-cycles that are one three-cycle “apart.” That is, three-cycles that share exactly one vertex. The \(S\)-polynomial \(S(x_i, x_{i+1}, x_{i+2}, x_{i+2} x_{i+3} x_{i+4})\) corresponds to the syzygy

\[
\begin{pmatrix}
0 \\
\vdots \\
0 \\
x_{i+3} x_{i+4} \\
x_i x_{i+1} \\
0 \\
\vdots \\
0
\end{pmatrix}
\]

where \(-x_{i+3} x_{i+4}\) is the \(i\)th component of the syzygy. However,

\[
\begin{pmatrix}
0 \\
\vdots \\
0 \\
-x_{i+2} x_{i+3} x_{i+4} \\
0 \\
x_i x_{i+1} x_{i+2} \\
0 \\
\vdots \\
0
\end{pmatrix}
= x_{i+2} x_{i+4} x_i + x_{i+2} x_{i+4} - x_{i+4} x_{i+1}
\]

That is,

\[
S(x_i, x_{i+1}, x_{i+2}, x_{i+2} x_{i+3} x_{i+4}) = x_{i+4} S(x_i, x_{i+1}, x_{i+2}, x_{i+1} x_{i+2} x_{i+3}) + x_i S(x_{i+1} x_{i+2} x_{i+3}, x_{i+2} x_{i+3} x_{i+4}).
\]
Similarly, for $S(x_ix_{i+1}x_{i+2}, x_{i+3}x_{i+4}x_{i+5})$ where $i \in \{0, \ldots, n - 5\}$, we see that

$$
\begin{pmatrix}
0 \\
\vdots \\
0 \\
-x_{i+3}x_{i+4}x_{i+5} \\
0 \\
x_ix_{i+1}x_{i+2} \\
0 \\
\vdots \\
0
\end{pmatrix}
= x_{i+4}x_{i+5}
\begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix}
= x_{i+4}x_{i+5}
\begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix}
= x_{i+4}x_{i+5}
\begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix}.
$$

So

$$
S(x_ix_{i+1}x_{i+2}, x_{i+3}x_{i+4}x_{i+5}) = x_{i+4}x_{i+5}S(x_ix_{i+1}x_{i+2}, x_{i+1}x_{i+2}x_{i+3})
\times
x_ix_{i+5}S(x_{i+1}x_{i+2}x_{i+3}, x_{i+2}x_{i+3}x_{i+4})
\times
x_ix_{i+1}S(x_{i+2}x_{i+3}x_{i+4}, x_{i+3}x_{i+4}x_{i+5}).
$$

Thus when $k = i + 1$, $S(x_ix_{i+1}x_{i+2}, x_kx_{k+1}x_{k+2})$ is a linear syzygy, when $k = i + 2$, $S(x_ix_{i+1}x_{i+2}, x_kx_{k+1}x_{k+2})$ is a linear combination of two linear syzygies, and when $k = i + 3$, $S(x_ix_{i+1}x_{i+2}, x_kx_{k+1}x_{k+2})$ is a linear combination of three linear syzygies.

In particular this shows that $\beta_{2,5} = 0$, as there are no quadratic syzygies in $F_2$. \hfill \square

The details of the above proof are perhaps best illustrated with an example.

**Example 15.** Let $n = 7$. Then the set of linear syzygies on $F_2$ is given by

$$
\left\{
\begin{pmatrix}
-x_3 \\
x_0 \\
0 \\
0
\end{pmatrix}
\begin{pmatrix}
0 \\
-x_4 \\
0 \\
0
\end{pmatrix}
\begin{pmatrix}
0 \\
x_1 \\
0 \\
0
\end{pmatrix}
\begin{pmatrix}
0 \\
-x_5 \\
x_2 \\
0
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
x_3 \\
-x_6
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
0 \\
x_4
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}
\right\}.
$$

Now consider the syzygy given by $S(x_0x_1x_2, x_2x_3x_4)$, which corresponds to two three-cycles that are exact two three-cycle “away” from
each other:
\[
\begin{pmatrix}
-x_3x_4 \\
0 \\
x_0x_1 \\
0 \\
0 \\
0
\end{pmatrix}.
\]

Note that
\[
\begin{pmatrix}
-x_3x_4 \\
0 \\
x_0x_1 \\
0 \\
0 \\
0
\end{pmatrix} = x_4 \begin{pmatrix}
-x_3 \\
x_0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix} + x_1 \begin{pmatrix}
0 \\
x_1 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}.
\]

Similarly, consider a syzygy corresponding to two three-cycles that are exactly three three-cycles apart, such as that given by \(S(x_0x_1x_2, x_3x_4x_5)\):
\[
\begin{pmatrix}
-x_3x_4x_5 \\
0 \\
0 \\
x_0x_1x_2 \\
0 \\
0
\end{pmatrix}.
\]

This can be written as
\[
\begin{pmatrix}
-x_3x_4x_5 \\
0 \\
0 \\
x_0x_1x_2 \\
0 \\
0
\end{pmatrix} = x_4x_5 \begin{pmatrix}
-x_3 \\
x_0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix} + x_0x_5 \begin{pmatrix}
0 \\
x_1 \\
0 \\
0 \\
0 \\
0
\end{pmatrix} + x_0x_1 \begin{pmatrix}
-x_5 \\
x_2 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}.
\]

We end with some further conjectures about the betti diagrams of these chordal graphs.

**Conjecture 16.** Let \(I^{(2)}_n\) be the secant ideal associated to the chordal graph \(G_n\). Then in the betti diagram of \(I^{(2)}_n\), we have
\[
\beta_{2,6} = \frac{1}{2}(n - 5)(n - 4)
\]
for \(n \geq 6\).

This entry in the betti diagram corresponds to the cubic syzygies in \(F_2\). We believe that these syzygies are generated by precisely the \(S\)-polynomials of three cycles that are more than three three-cycles.
“apart.” In general, there are $1 + 2 + \cdots + (n - 5) = \frac{1}{2}(n - 5)(n - 4)$ such syzygies. What remains to be shown, however, is that this set minimally generates the syzygy module. One possible proof strategy here is induction.

Under the base case of $n = 6$, there is exactly one such cubic syzygy. It is relatively easy to see that this syzygy cannot be written as a linear combination of linear syzygies. If it were possible to do so, we would expect

$$
\begin{pmatrix}
-x_5 x_6 x_7 \\
0 \\
0 \\
0 \\
x_0 x_1 x_2
\end{pmatrix}
= m_1
\begin{pmatrix}
-x_3 \\
x_0 \\
0 \\
0 \\
x_0 x_1 x_2
\end{pmatrix}
+ m_2
\begin{pmatrix}
0 \\
-x_4 \\
x_1 \\
0 \\
x_0 x_1 x_2
\end{pmatrix}
+ m_3
\begin{pmatrix}
0 \\
0 \\
-x_5 \\
x_2 \\
x_0 x_1 x_2
\end{pmatrix}
+ m_4
\begin{pmatrix}
0 \\
0 \\
0 \\
-x_6 \\
x_3 \\
0 \\
0 \\
x_0 x_1 x_2
\end{pmatrix}
+ m_5
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
x_0 x_1 x_2
\end{pmatrix}
$$

for some monomials $m_1, \ldots, m_5 \in S$. Notice that the $m_i$’s must all be of degree two and that it is not possible to have $m_5 x_4 = x_0 x_1 x_2$.

Our inductive hypothesis would assume that it is true that for some $n$, each of these cubic syzygies cannot be written as a linear combination of the linear syzygies and the other cubic syzygies. We could then apply the inductive hypothesis to $n + 1$, reducing the cubic syzygies under consideration to only those whose last component is nonzero.

We present an additional conjecture for which we have yet to develop a proof strategy.

**Conjecture 17.** Suppose $n \geq 5$. Then

$$\beta_{3,7} = (n - 5)^2.$$ 

Moreover, if $n \geq 6$,

$$\beta_{4,8} = \frac{1}{2}(n - 5)(n - 6).$$ 

**References**


