Igusa local zeta functions and $p$-adic analysis

Newton polyhedra and degenerate polynomials

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An introduction to $p$-adic valuation...

Given a number $a \in \mathbb{Q}$, the $p$-adic absolute value of $a$, denoted $|a|_p$, is defined as

$$|a|_p = \begin{cases} p^{-\text{ord}_p(a)} & \text{if } a \neq 0 \\ 0 & \text{if } a = 0. \end{cases}$$
Introduction

An introduction to $p$-adic valuation...

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- $\text{ord}_5\left(\frac{1}{25}\right) = -2 \Rightarrow \left|\frac{1}{25}\right|_5 = 5^2 = 25$

- $\text{ord}_3(18) = 2 \Rightarrow |18|_3 = 3^{-2} = \frac{1}{9}$
\( p \)-adic numbers

- The field of all \( p \)-adic numbers, \( \mathbb{Q}_p \), is defined as all equivalence classes of \( p \)-adic Cauchy sequences. A sequence \( \{x_i\} \) is Cauchy if for all \( \epsilon \) there exists \( N \in \mathbb{N} \) such that if \( m, n > N \), then 
\[ |x_n - x_m| < \epsilon. \]
$p$-adic numbers

- The field of all $p$-adic numbers, $\mathbb{Q}_p$, is defined as all equivalence classes of $p$-adic Cauchy sequences. A sequence \( \{x_i\} \) is Cauchy if for all $\epsilon$ there exists $N \in \mathbb{N}$ such that if $m, n > N$, then $|x_n - x_m| < \epsilon$.

- The ring of $p$-adic integers, $\mathbb{Z}_p$, is composed of all $p$-adic numbers $a \in \mathbb{Q}_p$ with $|a|_p \leq 1$.

Every $p$-adic integer $a$ has the form

$$a = a_0 + pa_1 + p^2a_2 + \ldots + p^ma_m + \ldots$$

for some $m \in \mathbb{Z}$. 
**$p$-adic numbers**

The *units* in $\mathbb{Z}_p$ are all $p$-adic integers $a$ with

$$|a|_p = 1.$$  

i.e. $p$-adic integers of the form

$$a = a_0 + pa_1 + \ldots + p^m a_m + \ldots$$

with $a_0 \neq 0$. 
Topology

The topology of $\mathbb{Z}_p$, for $p = 5$:

Note that every point in an open ball is a center of that ball.
Igusa local zeta function

The Igusa local zeta function associated to a polynomial $f(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$ is defined as

$$Z(s) = \int_{\mathbb{Z}_p^n} |f(x_1, \ldots, x_n)|_p^s \, dx_1 \ldots dx_n,$$

$s \in \mathbb{C}$, $Re(s) > 0$.

We use the convention $t = p^{-s}$. 
Stationary Phase Formula

\[ Z(s) = (p^n - |N_0|)p^{-n} + (|N_0| - |S|)p^{-n}t \left( \frac{1 - p^{-1}}{1 - p^{-1}t} \right) \]

\[ + \sum_{\alpha \in S} \int_{\alpha + p\mathbb{Z}_p^n} |f(x_1, \ldots, x_n)|^s \, dx_1 \ldots dx_n \]

where

\[ N_0 = \{(x_1, \ldots, x_n) \in \mathbb{F}_p^n \mid f(x_1, \ldots, x_n) \equiv 0 \pmod{p}\} \]

and

\[ S = \{(x_1, \ldots, x_n) \in N_0 \mid \frac{\partial f}{\partial x_i}(x) \equiv 0 \pmod{p}, 1 \leq i \leq n\}. \]
Ex. 1 - \( f(x) = x \)

\[
N_0 = \{x \mid x \equiv 0 \pmod{p}\} \Rightarrow |N_0| = 1
\]

\[
S = \{x \in N_0 \mid \frac{\partial f}{\partial x}(x) \equiv 0 \pmod{p}\} = \emptyset \Rightarrow |S| = 0,
\]

so using SPF...

\[
Z(s) = (p - 1)p^{-1} + (1 - 0)p^{-1}t \left( \frac{1 - p^{-1}}{1 - p^{-1}t} \right)
\]

\[
= \frac{1 - p^{-1}}{1 - p^{-1}t}
\]
Ex. 2 - \( f(x, y, z) = (x - y)^2 + z \)

\[ N_0 = \{(x, y, -(x - y)^2)\} \Rightarrow |N_0| = p^2 \]

\[ S = \emptyset \Rightarrow |S| = 0 \]

Zeta function:

\[ Z(s) = (p^3 - p^2)p^{-3} + (p^2 - 0)p^{-3}t \left( \frac{1 - p^{-1}}{1 - p^{-1}t} \right) \]

\[ = \frac{1 - p^{-1}}{1 - p^{-1}t} \]
Support of $f$

Given a polynomial

$$f(x_1, \ldots, x_n) = \sum_{\omega \in \mathbb{N}^n} a_\omega x_1^{\omega_1} \cdots x_n^{\omega_n},$$

the support of $f$ is defined as

$$supp(f) = \{\omega \in \mathbb{N}^n \mid a_\omega \neq 0\}.$$

Ex)

$$f(x, y) = xy - x^5 + x^2y^3$$

$$supp(f) = \{(1, 1), (5, 0), (2, 3)\}$$
Newton polyhedron

The Newton polyhedron $\Gamma(f)$ of a polynomial $f(x_1, \ldots, x_n)$, $f(0)=0$, is the convex hull in $(\mathbb{R}^+)^n$ of the set

$$\bigcup_{\omega \in \text{supp}(f)} \omega + (\mathbb{R}^+)^n.$$ 

$$f(x, y) = xy - x^5 + x^2 y^3$$
Faces and associated cones

- A face $\tau$ of $\Gamma(f)$ is the intersection of $\Gamma(f)$ with a supporting hyperplane that does not intersect the interior of $\Gamma(f)$. A facet is a face of dimension $n - 1$. 
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- A face $\tau$ of $\Gamma(f)$ is the intersection of $\Gamma(f)$ with a supporting hyperplane that does not intersect the interior of $\Gamma(f)$. A facet is a face of dimension $n - 1$.
- The cone associated to a facet $\tau$ is the normal vector to $\tau$. The cone for a face that is not a facet is the span of the cones for all facets containing the face.
Degeneracy

Given a polynomial $f(x_1, \ldots, x_n)$, the polynomial $f_{\tau}$ is composed of the terms of $f$ whose support is equal to $\text{supp}(f) \cap \tau$. 
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Given a polynomial $f(x_1, \ldots, x_n)$, the polynomial $f_\tau$ is composed of the terms of $f$ whose support is equal to $\text{supp}(f) \cap \tau$.

A polynomial $f(x_1, \ldots, x_n)$ is non-degenerate with respect to all faces of its Newton polyhedron if the system

$$ \begin{cases} f_\tau(x_1, \ldots, x_n) \equiv 0 \pmod{p} \\ \frac{\partial f_\tau}{\partial x_i}(x) \equiv 0 \pmod{p} \end{cases} $$

has no non-zero solutions.
\( \sigma(\mathbf{k}) \) and \( m(\mathbf{k}) \)

For an \( n \)-vector \( \mathbf{k} \),

\[
\sigma(\mathbf{k}) := \sum_{i=1}^{n} k_i
\]

and

\[
m(\mathbf{k}) := \inf_{\mathbf{x} \in \Gamma(f)} \{ \mathbf{k} \cdot \mathbf{x} \}.
\]
Non-degenerate polynomials

For a polynomial $f(x_1, \ldots, x_n)$ that is nondegenerate with respect to all faces of its Newton polyhedron, the Igusa local zeta function associated to $f$ is

$$Z(s) = \sum_{\tau \in \Gamma(f)} L_{\tau} S_{\Delta_{\tau}},$$

where

$$L_{\tau} = p^{-n} \left( (p - 1)^n - p|N_{\tau}| \left( \frac{p^s - 1}{p^{s+1} - 1} \right) \right),$$

$$N_{\tau} = \{(x_1, \ldots, x_n) \in (\mathbb{F}_p^*)^n \mid f_\tau(x_1, \ldots, x_n) \equiv 0 \pmod{p} \},$$

$$S_{\Delta_{\tau}} = \sum_{k} p^{-\sigma(k) + m(k)s}.$$
Degenerate polynomials

For a polynomial which is degenerate with respect to some faces of its Newton polyhedron, $S_{\Delta \tau}$ doesn’t change, but $L_{\tau}$ does.

$$\overline{L}_{\tau} = p^{-n}((p-1)^n - |N_{\tau}|) + (|N_{\tau}| - |S|) p^{-n}t \left( \frac{1 - p^{-1}}{1 - p^{-1}t} \right).$$

Note that for all faces for which $f_{\tau}$ is non-degenerate, $L_{\tau}$ remains as in the original formula.
Degenerate polynomials 2

\[ Z(s) = \sum_{\tau \text{ nondeg.}} L_\tau S_{\Delta \tau} \]

\[ + \sum_{\tau \text{ deg.}} \left( L_\tau S_{\Delta \tau} + \sum_{k} \left( p^{-(\sigma(k)+m(k))s} \sum_{\alpha \in S} \int_{\alpha + p\mathbb{Z}_p^n} |f_\tau + p\tilde{f}|^s \, du_1 \ldots du_n \right) \right) \]

where \( f_\tau + p\tilde{f} = p^{m(k)} f \).
Example

Recall the polynomials from earlier:

1. \( f(x) = x \)
2. \( f(x, y, z) = (x - y)^2 + z \)
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2. \( f(x, y, z) = (x - y)^2 + z \)

\[
Z(s) = \frac{1 - p^{-1}}{1 - p^{-1}t}
\]

for both polynomials, but (1) is non-degenerate while (2) is degenerate!
Future research

- Compare polynomials, both non-degenerate and degenerate, which have the same ILZF.
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- Compare polynomials, both non-degenerate and degenerate, which have the same ILZF.
- Find classes of polynomials for which more can be said about the integral over the singular points in the Newton polyhedron method.
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