1. Consider the sequence defined by the recurrence relation:

\[ a_1 = 2, \quad a_2 = 4, \quad \text{and} \quad a_n = \frac{a_{n-1}}{a_{n-2}} \quad \text{for} \quad n > 2. \]

a. Determine the values of \( a_3, a_4, a_5, \) and \( a_6 \), and plot the points in the graph of this sequence for numbers in the domain of the sequence 1, 2, 3, 4, 5, and 6.

\[
a_3 = \frac{a_3-1}{a_3-2} = \frac{a_2}{a_1} = \frac{4}{2} = 2, \quad a_4 = \frac{a_4-1}{a_4-2} = \frac{a_3}{a_2} = \frac{2}{4} = \frac{1}{2}, \quad a_5 = \frac{a_5-1}{a_5-2} = \frac{a_4}{a_3} = \frac{\frac{1}{2}}{2} = \frac{1}{4}, \quad a_6 = \frac{a_6-1}{a_6-2} = \frac{a_5}{a_4} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}.
\]

The graph \((n, a_n)\) is plotted below for \( n = 1, 2, 3, 4, 5, \) and 6.
b. Determine whether \( \lim_{n \to \infty} a_n \) exists, and justify your claim.

If we compute terms past the ones determined in part a, we get
\[
a_7 = \frac{a_{7-1}}{a_{7-2}} = \frac{a_6}{a_5} = \frac{1}{2} = 2 \quad \text{and} \quad a_8 = \frac{a_{8-1}}{a_{8-2}} = \frac{a_7}{a_6} = \frac{2}{1} = 4 ,
\]
the two numbers starting off the sequence. So, the same progression, 2, 4, 2, \( \frac{1}{2} \), \( \frac{1}{4} \), \( \frac{1}{2} \) will just be repeated \textit{ad infinitum}. Thus, the sequence can never get arbitrarily close to any one number and can have no limit.

2. For each of the series below explain why it must converge. For series a, determine the number to which it converges. For series b, determine an approximation for the number to which it converges, so that the error in the approximation is no larger than 0.005.

a. \[ 2 - \frac{6}{5} + \frac{18}{25} - \frac{54}{125} + \frac{162}{625} - \frac{486}{3125} + \ldots \]
Each term in the series is obtained from the previous one by multiplying by the constant \( -\frac{3}{5} \), so the series is geometric with constant ratio that has absolute value less than 1. Thus the series converges to the number \( \frac{2}{1 - (-\frac{3}{5})} = \frac{2}{8} = \frac{5}{4} \), according to the formula for the sum of a convergent geometric series: first term divided by \( (1 - \text{constant ratio}) \).

b. 1.9191191119111191111119 …

To show that this infinite decimal is a convergent infinite series, write it as \[ 1 + \frac{9}{10} + \frac{1}{100} + \frac{9}{10^3} + \frac{1}{10^4} + \frac{9}{10^5} + \frac{1}{10^6} + \frac{9}{10^7} + \frac{1}{10^8} + \ldots \] This sequence of partial sums of this series is increasing, because each term of the series is positive, and the sequence of partial sums never
exceeds 1.92, so it is bounded. Therefore, by the Monotone Bounded Convergence Theorem the series converges. (The series converges because its sequence of partial sums has a limit.)

To get an estimate, first get an idea of what’s going by just making a one-decimal place approximation 1.9 for the infinite decimal. The error in making this estimate is the difference: 1.9191191119111911191119… - 1.9 = 0.019119111911119…, which is less than 0.02, not quite as small as required. If we round off to two decimal places, 1.92, the error is a little trickier to compute, but it can’t be any larger than the difference between 1.92 (which is clearly larger than the infinite decimal) and 1.919 (which is clearly smaller than the infinite decimal). This difference is 0.001, which is within the required bound for the error.

3. Use the Comparison Test in part a below to determine whether that series converges or diverges. In part b also determine whether the series converges or diverges and, if convergent, what the sum of the series is.

a. \( \sum_{n=1}^{\infty} \frac{50e^{-2n}}{n^{-1}} \)

Since \( \sum_{n=1}^{\infty} \frac{50e^{-2n}}{n^{-1}} = 50 \sum_{n=1}^{\infty} \frac{n}{e^{2n}} \), compare \( \sum_{n=1}^{\infty} \frac{n}{e^{2n}} \) to the convergent geometric series \( \sum_{n=1}^{\infty} \frac{1}{e^n} \). (This latter series is convergent, since the constant ratio is \( \frac{1}{e} \), which is less than 1 in absolute value.) If we can show that \( \frac{n}{e^{2n}} < \frac{1}{e^n} \) for \( n = 1, 2, 3, 4, \ldots \), then we’ll know that \( \sum_{n=1}^{\infty} \frac{n}{e^{2n}} \) is convergent by the Comparison Test, and then we’ll have what we need: that \( \sum_{n=1}^{\infty} \frac{50e^{-2n}}{n^{-1}} = 50 \sum_{n=1}^{\infty} \frac{n}{e^{2n}} \) is convergent by part i of the Theorem at the top of page 719 that says that multiplying each term of a convergent series by one constant preserves the
converge. So now for \( \frac{n}{e^{2n}} < \frac{1}{e^n} \), which is equivalent to \( n < e^n \): We appeal to ideas from Calculus I.

1. The function \( x \) is increasing at a constant rate (zero 2\(^{nd}\) derivative) while the function \( e^x \) is increasing at an increasing rate (2\(^{nd}\) derivative positive).
2. The derivatives (rates of increase) when \( x=1 \) for these two functions are 1 (for \( x \)) and \( e \) (for \( e^x \)).
3. So, the function \( e^x \) is itself greater than \( x \) when \( x=1 \), is increasing at a greater rate than \( x \) is increasing when \( x=1 \), and continues to increase at an increasing rate of increase while \( x \) increases at its constant rate, thus insuring that \( x < e^x \) for all \( x \) greater than 1.

b. \[ \sum_{n=1}^{\infty} \frac{100(2)^{n+2}}{3^n} \]

The series can be written as \[ \sum_{n=1}^{\infty} \frac{100(2)^{2}(2)^n}{3^n} = \sum_{n=1}^{\infty} \frac{400(2)^{n}}{3^n} \], which is a convergent geometric series whose ratio is \( \frac{2}{3} \) and whose first term is \( \frac{800}{3} \). Using the formula for the sum of a convergent series, the series converges to \( \frac{800}{1 - \frac{2}{3}} = 800 \).

4.

a. Give an example of an infinite series \( \sum_{n=1}^{\infty} a_n \), other than the Harmonic series, for which \( \lim_{n \to \infty} a_n = 0 \) but \( \sum_{n=1}^{\infty} a_n \) diverges. Explain why your infinite series must diverge. If you are unable to construct such an example, tell what you know about infinite series and what it means to say that an infinite series diverges.
Since the Harmonic series has the desired property but is disallowed, we might think of constructing something similar. Thinking about the Comparison Test leads to trying to make an nth term that is bigger than \( \frac{1}{n} \), but still approaches 0. A simple way of doing this is to take the series whose nth term is \( \frac{2}{n} \). Then by the Comparison Test the series \( \sum_{n=1}^{\infty} \frac{2}{n} \) must diverge, but the sequence of its terms approaches 0.

We could also do something like: Start with \( \frac{1}{n} = \frac{n}{n^2} \) and make it larger by adding to the numerator and subtracting from the denominator, say \( \frac{n+2}{n^2-1} \). Then \( \frac{n+2}{n^2-1} > \frac{1}{n} \), since \( n^2 + 2n > n^2 - 1 \); and

\[
\lim_{n \to \infty} \frac{n+2}{n^2-1} = \lim_{n \to \infty} \left( \frac{\frac{n+2}{n}}{\frac{n^2-1}{n}} \right) = \lim_{n \to \infty} \left( \frac{\frac{1}{n} + \frac{2}{n}}{\frac{1}{n} - \frac{1}{n^2}} \right) = 0.
\]

By the Comparison Test, \( \sum_{n=2}^{\infty} \frac{n+2}{n^2-1} \) must diverge. Note that we need to avoid a zero denominator, so the index n starts at 2.

b. Use the blocks of increasing lengths of powers of 2 shown on page 717 of Calculus to determine a number N so that the Nth partial sum of the Harmonic series exceeds 100. Be sure that you show carefully how you obtain your value of N.

Let’s take a look, using the powers of 2 length blocks, at the first, say, 64 terms in order to get an idea of what’s going on: this 64th partial sum is

\[
1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + (\frac{1}{9} + \ldots + \frac{1}{16}) + (\frac{1}{17} + \ldots + \frac{1}{2^5})
\]

\[
+ \left( \frac{1}{33} + \ldots + \frac{1}{2^6} \right) > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{6}{2}
\]

So, each of the blocks contributes a sum no less than \( \frac{1}{2} \) and for this case there are 6 blocks plus the initial value 1. To see how many blocks we need, we look at the relation, \( 1 + \frac{x}{2} = 100 \), solve for x
and get that we need 198 blocks, each contributing more than \( \frac{1}{2} \) to get the partial sum to exceed 100. Now we examine the pattern of blocks and see that the first block, of length 1, ends with \( \frac{1}{2} \), the second block, of length 2, ends with \( \frac{1}{2^2} \), the third block, of length 4, ends with \( \frac{1}{2^3} \), and it’s clear that the 198\(^{th} \) block will end with \( \frac{1}{2^{198}} \), which, being the \( 2^{198} \)th term of the series, means that \( 2^{198} \) works as a number N so that the Nth partial sum exceeds 100.

In terms of powers of 10, using the approximation \( 2^{10} = 1,024 \approx 10^3 \), \( 2^{200} = 2^{(10)(20)} = (2^{10})^{20} \approx (10^3)^{20} = 10^{60} \), which kind of makes you wonder if this series really can diverge if it takes \( 10^{60} \) terms to add up to just 100. It’s a long way to “infinity!”