1. In-Class. Show that the Alternating Series Test applies to the infinite series \( \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1}{2}\right)^n \) and that the test proves that this series converges.

The series \( \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1}{2}\right)^n \) alternates because i) the exponent for \(-1\) alternates between an even number and an odd number as that exponent, \(n+1\), runs through the positive integers 2, 3, 4, \ldots ; ii) the absolute values of the terms, \(\left(\frac{1}{2}\right)^n\), decrease:

\[
(2)^n < (2)^{n+2n+1} = (2)^{(n+1)^2}, \text{ so } \left(\frac{1}{2}\right)^n > \left(\frac{1}{2}\right)^{(n+1)^2}; \text{ and iii) the limit of the absolute value of the nth term } \left(\frac{1}{2}\right)^n \text{ is zero: Now } \left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^{n(n)} = \left(\frac{1}{2^n}\right)^n \text{ and } 0 < \frac{1}{2^n} < 1, \text{ so } \lim_{n \to \infty} \left(\frac{1}{2}\right)^n = \lim_{n \to \infty} \left(\frac{1}{2^n}\right)^n = 0. \text{ Therefore by the Alt Series Test, the series } \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1}{2}\right)^n \text{ converges.}
\]

a. Consider the infinite series \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \). Determine an upper bound for the error in approximating the sum of this series by the sum:

\[
1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots - \frac{1}{198} + \frac{1}{199}. \text{ (Assume that there is no error in computing the finite sum.)}
\]

Since the series is (i) alternating, (ii) decreasing, and (iii) terms approaching zero, the error can be no larger than the absolute value of the first omitted term, which is \(1/100\).
b. Determine a number of terms – call it N for now - in the infinite series \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5} \) that will assure that if the sum of this infinite series is approximated by the first N terms, then the error in this approximation is no more than 0.0000000001.

To use the theorem on a bound for error of an alternating series, we need to note that the series is alternating as a consequence of the exponent for –1 alternating between an even number and an odd number as n+1 increases by 1, and the sequence of terms, \( \frac{1}{n^5} \), of the series is clearly decreasing and has limit zero as n approaches infinity.

Applying the theorem, we need N so the the absolute value of the first term omitted, \( \frac{1}{(N+1)^5} \), is less than 0.0000000001/. So, we solve the inequality \( \frac{1}{(N+1)^5} < \frac{1}{10^{10}} \), which is equivalent to \( 10^{10} < (N+1)^5 \), which is, in turn, equivalent to \( 10^2 < N+1 \). Thus N=100 will suffice as the number of terms in the series to add up to be sure you get an estimate for the infinite sum that is within 0.0000000001 of the sum for the infinite series.

2. a. Use the Ratio Test to determine whether the series \( \sum_{n=1}^{\infty} \frac{n^2 7^{n+2}}{2^{3n-1}} \) converges or diverges. Explain your reasoning.

The absolute value of the ratio of the (n+1)st term to the nth term

\[
\left| \frac{(n+1)^2 (7)^{n+3}}{n^2 (7)^{n+2}} \right| = \frac{(n+1)^2 (7)(7)^{n+2}}{n^2 (7)^{n+2}} \frac{2^{3n+1}}{2^{3n-1}} = \frac{(n+1)^2 (7)^{n+3}}{n^2 (7)^{n+2}} \frac{2^{2n+1}}{2^{2n-1}} = \frac{(n+1)^2 (7)^{n+3}}{n^2 (7)^{n+2}} \frac{2^{2n+1}}{2^{2n-1}} = \frac{7}{2} = \frac{1 + \frac{1}{n}}{\frac{7}{8}}
\]

is greater than 1, so the series diverges.
which approaches $\frac{7}{8}$ as $n$ approaches infinity. Since this limit ratio is less than 1, the series converges by the Ratio Test.

b. Determine whether the series
\[
\sum_{n=1}^{\infty} \frac{(2)^n(n!)}{(5)(8)(11)(14)\cdots(3n+2)} = \frac{(2)}{(5)} + \frac{(2)^2(2!)}{(5)(8)} + \frac{(2)^3(3!)}{(5)(8)(11)} + \cdots
\]
converges or diverges and explain why it does so.

Using the Ratio Test, consider the ratio of the $(n+1)$st term of the series to the $n$th term:
\[
\frac{(2)^{n+1}(n+1)!}{(5)(8)(11)(14)\cdots(3[n+1]+2)} \cdot \frac{(2)^n(n!)}{(5)(8)(11)(14)\cdots(3n+2)} = \frac{(2)^{n+1}(n+1)!}{(5)(8)(11)(14)\cdots(3n+1)+2} \cdot \frac{(2)^n(n!)}{(5)(8)(11)(14)\cdots(3n+2)} = \frac{2(n+1)}{(3n+5)} = \frac{2 + \frac{2}{n}}{(3 + \frac{5}{n})},
\]
which approaches $\frac{2}{3}$ as $n$ approaches infinity.

Since this limit ratio is less than 1, the series converges by the Ratio Test.
3. a. Evaluate the indefinite integral $\int e^{\frac{-2}{x^2}} \, dx$ by making the substitution $u = \frac{-2}{x}$.

Letting $u = \frac{-2}{x}$, $du = \frac{2}{x^2} \, dx$ and $\frac{1}{2} \, du = \frac{1}{x^2} \, dx$. The integral then transforms:

$$\int e^{\frac{-2}{x^2}} \, dx = \int e^{u} \frac{2}{2} \, du = \frac{e^u}{2} + c = \frac{e^{\frac{-2}{x^2}}}{2} + c.$$  

To check:

$$\frac{d}{dx} \left( \frac{e^{\frac{-2}{x^2}}}{2} + c \right) = \frac{e^{\frac{-2}{x^2}}}{2} \frac{d}{dx} \left( \frac{-2}{x^2} \right) = \frac{e^{\frac{-2}{x^2}}}{2} \left( 2 \cdot 2x^{-3} \right) = \frac{e^{\frac{-2}{x^2}}}{x^3}$$  

by the chain rule.

b. This problem concerns the sequence obtained from the 3-step folding process (shown in class) to estimate $1/7$.

(i) Write a formula for the second term, $a_2$, in the sequence $(a_n)$ of estimates for $1/7$, given any guess (any estimate) $a_1$.

(ii) Write a recursion formula for the $n$th term in this sequence.

(iii) Write an explicit formula for the $n$th term in this sequence.

(iv) Show that this sequence converges and that its limit is the same as the limit of partial sums of a geometric series that converges to $1/7$.

We start with the result of the 3-fold process $a_2 = \frac{1 + a_1}{8} = \frac{1}{8} + \frac{a_1}{8}$.

Since the same 3 folds are used to get $a_n$ from $a_{n-1}$, we have the recursion formula $a_n = \frac{1}{8} + \frac{a_{n-1}}{8}$, for $n > 1$ and $a_1$ arbitrary. To get an explicit formula, we start calculating: We get

$$a_3 = \frac{1}{8} + \frac{a_2}{8} = \frac{1}{8} + \frac{\frac{1}{8} + \frac{a_1}{8}}{8} = \frac{1}{8} + \frac{1}{64} + \frac{a_1}{64},$$

and, similarly

$$a_4 = \frac{1}{8} + \frac{a_3}{8} = \frac{1}{8} + \frac{\frac{1}{8} + \frac{1}{64} + \frac{a_1}{64}}{8} = \frac{1}{8} + \frac{1}{64} + \frac{1}{512} + \frac{a_1}{512}.$$  

Each time, we add $\frac{1}{8}$ to $\frac{1}{8}$ of the immediately preceding term. Thus, we get the explicit
formulas for $a_n$: $a_n = \frac{1}{8} + \frac{1}{64} + \frac{1}{512} + \cdots + \frac{1}{8^{n-1}} + \frac{a_1}{8^{n-1}}$. Then

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \left( \frac{1}{8} + \frac{1}{64} + \frac{1}{512} + \cdots + \frac{1}{8^{n-1}} \right) + \lim_{n \to \infty} \left( \frac{a_1}{8^{n-1}} \right) =$$

$$\lim_{n \to \infty} \sum_{i=2}^{n} \frac{1}{8^{i-1}} = \sum_{i=2}^{\infty} \frac{1}{8^{i-1}} = \frac{\frac{1}{8}}{1 - \frac{1}{8}} = \frac{1}{7}, \text{ since } \lim_{n \to \infty} \sum_{i=2}^{n} \frac{1}{8^{i-1}} \text{ is the limit of the sequence of partial sums of the infinite convergent geometric series } \sum_{i=2}^{\infty} \frac{1}{8^{i-1}} \text{ (whose constant ratio is } \frac{1}{8}) \text{ and } \lim_{n \to \infty} \left( \frac{a_1}{8^{n-1}} \right) = 0.$$

4. a. Evaluate the definite integral $\int_{\frac{1}{2}}^{1} x \sin(\pi x) \, dx$ using Integration by Parts with $u = x$ and $dv = \sin(\pi x) \, dx$.

With $u = x$ and $dv = \sin(\pi x) \, dx$, $du = dx$ and $v = -\frac{1}{\pi} \cos(\pi x)$. The integration by parts formula, $\int u \, dv = uv - \int v \, du$, implies that

$$\int_{\frac{1}{2}}^{1} x \sin(\pi x) \, dx = x \left( -\frac{1}{\pi} \cos(\pi x) \right) \bigg|_{\frac{1}{2}}^{1} - \int_{\frac{1}{2}}^{1} -\frac{1}{\pi} \cos(\pi x) \, dx =$$

$$-\frac{1}{\pi} \cos(\pi) + \frac{1}{2} \left( -\frac{1}{\pi} \cos(\frac{\pi}{2}) \right) + \left[ \frac{1}{\pi^2} \sin(\pi x) \right]_{\frac{1}{2}}^{1} =$$

$$\frac{1}{\pi} + \frac{1}{\pi^2} \sin(\pi) - \frac{1}{\pi^2} \sin(\frac{\pi}{2}) = \frac{1}{\pi} - \frac{1}{\pi^2}$$

b. Evaluate the definite integral $\int_{1}^{2} \frac{2 \ln(x) + 1}{x^3} \, dx$. 

Let \( u = 2\ln x + 1 \) and \( dv = \frac{1}{x^3}dx \). Then \( \int \frac{2\ln x + 1}{x^3}dx = \int udv \). Since \( du = \frac{2}{x}dx \) and \( v = \frac{x^2}{2} \), the integration by parts formula says that

\[
\int_{1}^{2} \frac{2\ln x + 1}{x^3}dx = [(2\ln x + 1)(\frac{x^2}{2})]\bigg|_{1}^{2} - \int_{1}^{2} (\frac{x^2}{2}) \cdot \frac{2}{x}dx =
\]

\[
[(2\ln 2 + 1)(\frac{2}{2})] - [(2\ln 1 + 1)(\frac{1}{2})] + \int_{1}^{2} x^{-3}dx =
\]

\[
[(2\ln 2 + 1)(\frac{-1}{8})] + [\frac{1}{2}] + (\frac{x^{-2}}{-2})\bigg|_{1}^{2} = \frac{-\ln 2}{4} + \frac{3}{8} + (\frac{2^{-2}}{-2}) - (\frac{1^{-2}}{-2}) =
\]

\[
\frac{-\ln 2}{4} + \frac{3}{8} + \frac{-1}{8} + \frac{1}{2} = \frac{-\ln 2}{4} + \frac{3}{4} = \frac{3 - \ln 2}{4}
\]