Cooling Brownies

We had reached the point in class on Wed., 12/7, where

$$\frac{1}{298} (\ln(y - 298) - \ln(y)) = kt + \frac{1}{298} (\ln(390 - 298) - \ln(390)).$$

The above result is based on using Newton's Law of Cooling:

$$\frac{dy}{dt} = ky(y - 298),$$

with ambient temperature $298^0 K$ and $y(0) = 390^0 K$. (We separated variables, used Integral Formula #26, and used $y(0) = 390^0 K$ to solve for the constant of integration.) Rather than continue to use the messy specific numbers,

I'll use $T =$ Ambient Temperature in place of $298^0 K$,

$T_0 = y(0) =$ Initial Temperature of the object in place of 390, and

and $T_1 = y(t_1)$ for some known (or estimated) temperature $T_1$ at some known time $t_1$ in place of $310 = y(30)$. With these parameters in place, we have:

$$\frac{1}{T} (\ln(y - T) - \ln(y)) = kt + \frac{1}{T} (\ln(T_0 - T) - \ln(T_0)),$$

or

$$(\ln(y - T) - \ln(y)) = kTt + (\ln(T_0 - T) - \ln(T_0))$$

Now use $T_1 = y(t_1)$ to get

$$(\ln(T_1 - T) - \ln(T_1)) = kTt_1 + (\ln(T_0 - T) - \ln(T_0)).$$

Using properties of log's and solving for $kT$,

$$kT = \frac{1}{t_1} (\ln(\frac{T_1 - T}{T_1}) - \ln(\frac{T_0 - T}{T_0})) = \frac{1}{t_1} \ln[(\frac{T_1 - T}{T_1})(\frac{T_0}{T_0 - T})].$$
Going back to \((\ln(y - T) - \ln(y)) = kT + (\ln(T_0 - T) - \ln(T_0))\),

\[
\ln\left(\frac{y - T}{y}\right) = kT + (\ln\left(\frac{T_0 - T}{T_0}\right)) = \frac{T}{t_1} \ln\left([\frac{T_1 - T}{T_1}\left(\frac{T_0}{T_0 - T}\right)]^{t_1}\left(\frac{T_0 - T}{T_0}\right)\right) + (\ln\left(\frac{T_0 - T}{T_0}\right))
\]

\[
\Rightarrow \frac{y - T}{y} = [\frac{T_1 - T}{T_1}\left(\frac{T_0}{T_0 - T}\right)]^{t_1}\left(\frac{T_0 - T}{T_0}\right). \text{ Finally, solving for } y,
\]

\[
y = \frac{T}{1 - \left(\frac{T_0 - T}{T_0}\right)[\left(\frac{T_1 - T}{T_1}\right)(\frac{T_0}{T_0 - T})]^{t_1}}. \text{ To see what happens as } t \to \infty,
\]

look at the equivalences:

\[
\frac{T_1 - T}{T_1}\left(\frac{T_0}{T_0 - T}\right) < 1 \iff (T_1 - T)T_0 < T_1(T_0 - T)
\]

\[
\iff (T_1 - T)T_0 < T_1(T_0 - T) \iff (T_1 - T)T_0 > T_1 \iff \text{The temperature at a time } t_1 > 0
\]

is less than the initial temperature. What this means is that

\[
\frac{T_1 - T}{T_1}\left(\frac{T_0}{T_0 - T}\right) < 1, \quad \lim_{t \to \infty}[\left(\frac{T_1 - T}{T_1}\right)(\frac{T_0}{T_0 - T})]^{t_1} = 0 \text{ and}
\]

\[
\lim_{t \to \infty} \frac{T}{1 - \left(\frac{T_0 - T}{T_0}\right)[\left(\frac{T_1 - T}{T_1}\right)(\frac{T_0}{T_0 - T})]^{t_1}} = T.
\]

In the case of our brownies,

\[
y = \frac{298}{1 - \left(\frac{390 - 298}{390}\right)[\left(\frac{310 - 298}{310}\right)(\frac{390}{390 - 298})]^{\frac{t}{30}}}
\]

\[
= \frac{298}{1 - \left(\frac{92}{390}\right)[\left(\frac{12}{310}\right)(\frac{390}{92})]^{\frac{t}{30}} \approx \frac{298}{1 - (0.2359)(0.1641)^{\frac{t}{30}}}}
\]