9.3: #2 To decide whether \( \sum_{n=1}^{\infty} \frac{n}{n^2 + 1} \) converges, consider the function

\[ f(x) = \frac{x}{x^2 + 1} \]

and the improper integral \( \int_{0}^{\infty} \frac{x}{x^2 + 1} \, dx \). To determine

\[ \int_{0}^{\infty} \frac{x}{x^2 + 1} \, dx, \] let \( u = x^2 + 1 \). Then \( du = 2x \, dx \) and

\[ \int_{0}^{\infty} \frac{x}{x^2 + 1} \, dx = \int_{0}^{\infty} \frac{1}{2u} \, du = \frac{1}{2} \ln |u| + c = \frac{1}{2} \ln(x^2 + 1) + c. \] Thus

\[ \int_{0}^{\infty} \frac{x}{x^2 + 1} \, dx = \lim_{b \to \infty} \int_{0}^{b} \frac{x}{x^2 + 1} \, dx = \lim_{b \to \infty} \frac{1}{2} \ln(b^2 + 1) \bigg|_{0}^{b} = \lim_{b \to \infty} \frac{1}{2} \ln(b) = \infty, \]

and hence, by the Integral Test, the series \( \sum_{n=1}^{\infty} \frac{n}{n^2 + 1} \) also diverges.

#4 To decide whether \( \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} \) converges, consider the function

\[ f(x) = \frac{1}{x(\ln x)^2} \]

and the improper integral \( \int_{2}^{\infty} \frac{1}{x(\ln x)^2} \, dx \). To determine

\[ \int_{2}^{\infty} \frac{1}{x(\ln x)^2} \, dx, \] let \( u = \ln x \). Then \( du = \frac{1}{x} \, dx \)

and

\[ \int_{2}^{\infty} \frac{1}{x(\ln x)^2} \, dx = \int_{2}^{\infty} \frac{1}{u^2} \, du = -u^{-1} + c = -\frac{1}{u} + c = -\frac{1}{\ln x} + c. \] Thus

\[ \int_{2}^{\infty} \frac{1}{x(\ln x)^2} \, dx \]
\[
\lim_{b \to \infty} \left[ \frac{-1}{\ln x} \right]_2^b = \lim_{b \to \infty} \frac{-1}{\ln(b)} + \lim_{b \to \infty} \frac{1}{\ln(2)} = 0 + \frac{1}{\ln(2)} = \frac{1}{\ln(2)}, \text{ and hence, by the Integral Test, the series } \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} \text{ also converges.}
\]

#34a To decide whether \( \sum_{n=2}^{\infty} \frac{1}{n(\ln n)} \) converges, consider the function

\[ f(x) = \frac{1}{x(\ln x)} \]

and the improper integral \( \int_2^{\infty} \frac{1}{x(\ln x)} \, dx \). To determine \( \int_2^{\infty} \frac{1}{x(\ln x)} \, dx \), let \( u = \ln x \). Then \( du = \frac{1}{x} \, dx \)

and

\[ \int_2^{\infty} \frac{1}{x(\ln x)} \, dx = \int_{\ln 2}^{\infty} \frac{1}{u} \, du = \ln |u| + c = \ln |\ln x| + c. \]

Thus

\[ \int_2^{\infty} \frac{1}{x(\ln x)} \, dx = \lim_{b \to \infty} \int_2^{b} \frac{1}{x(\ln x)} \, dx = \lim_{b \to \infty} \left[ \ln |\ln x| \right]_2^b = \infty, \text{ and hence, by the Integral Test, the series } \sum_{n=2}^{\infty} \frac{1}{n(\ln n)} \text{ also diverges.} \]

7.3: \#2 \( \int x^5 \ln x \, dx = \frac{1}{6} x^6 \ln x - \frac{1}{6^2} x^6 + c \) by Table III \#13

\#6 To solve \( \int \sin w \cos^4 w \, dw \), following the guidance of Table IV \#23, let \( u = \cos w \). Then \( du = -\sin w \, dw \) and

\[ \int \sin w \cos^4 w \, dw = \int -u^4 \, du = \frac{-1}{5} u^5 + c = \frac{-1}{5} \cos^5 w + c. \]
#8 \int \frac{1}{3 + y^2} dy = \frac{1}{\sqrt{3}} \arctan \frac{y}{\sqrt{3}} + c, \text{ using Table V \#24 with } a = \sqrt{3}.

#10 To solve \int \frac{1}{9x^2 + 16} \, dx, \text{ you can use Table V \#24 again, but first let } u = 3x \text{ and } du = 3dx. \text{ Then } \int \frac{1}{9x^2 + 16} \, dx = \int \frac{1/3}{u^2 + 16} \, du. \text{ Now Formula \#24 applies and }

\int \frac{1}{9x^2 + 16} \, dx = \int \frac{1/3}{u^2 + 16} \, du \\
= \frac{1}{3} \arctan \frac{u}{4} + c = \frac{1}{12} \arctan \frac{3x}{4} + c.

Note. It is tempting to think that Formula \#24 applies directly to \int \frac{1}{9x^2 + 16} \, dx. \text{ What is important to note is that Formula \#24, }

\int \frac{1}{x^2 + a^2} \, dx = \frac{1}{a} \arctan \frac{x}{a} + c, \text{ has the form: } \int \frac{1}{(\text{glob})^2 + a^2} \, d(\text{glob}) = \frac{1}{a} \arctan \frac{\text{glob}}{a} + c \text{ and those "globs" have } \\
\text{to be exactly the same. More precisely, the variable with respect to which the integration is done must be the variable that is squared in the denominator of \#24. This matching of } d(\text{variable}) \text{ and } \text{variable}^2 \text{ explains why the substitution } u = 3x \text{ must be made.}

#20 (Started in class)

7.4: \#2

\frac{x + 1}{6x + x^2} = \frac{x + 1}{x(6 + x)} = A + \frac{B}{6 + x} = \frac{6A + 6x + Bx}{x(6 + x)} = \frac{6A + (6 + B)x}{x(6 + x)}
Comparing the numerators, it suffices to have 6A=1 and 6+B=1, or

\[ A = \frac{1}{6} \text{ and } B = -5. \]

Thus

\[ \frac{x+1}{6x+x^2} = \frac{1}{6} \frac{1}{x} + \frac{-5}{6+x}. \]

#8

\[
\frac{20}{25-x^2} = \frac{20}{(5+x)(5-x)} = \frac{A}{5+x} + \frac{B}{5-x} = \frac{5A-Ax+5B+Bx}{25-x^2}
\]

\[ = \frac{5A+5B+(-A+B)x}{25-x^2} \]

Comparing the numerators, it suffices to have

5A + 5B = 20 \text{ and } -A + B = 0, \text{ so } A = 2 \text{ and } B = 2 \text{ works and}

\[
\frac{20}{25-x^2} = \frac{2}{5+x} + \frac{2}{5-x}. \text{ Then}
\]

\[
\int \frac{20}{25-x^2} \, dx = \int \frac{2}{5+x} \, dx + \int \frac{2}{5-x} \, dx = 2 \ln |5+x| - 2 \ln |5-x| + c
\]

which also can be written \( \ln \left( \frac{5+x}{5-x} \right)^2 + c \)

#10 (Started in Class)