Calculus II – 02
Solutions to Pyramid Exam 1 – Take-Home Portion
Each problem part (whole problem, if it isn’t divided into parts) is equally weighted.

1. Write an expression for a sequence \((a_n)\) so that \(a_1 = 1/2, a_2 = 2/5, a_3 = 3/8, a_4 = 4/11,\) and \(a_5 = 5/14.\)

The numerators increase by 1, starting with 1, and denominators by 3, starting with 2, so \(a_n = \frac{n}{3n-1}\) will work. Note that this problem concerns only a sequence, not an infinite series.

b. For the sequence \((a_n)\) that you defined in part (a), what is \(\lim_{n \to \infty} a_n?\)

Since \(a_n = \frac{n}{3n-1} = \frac{1}{3 - \frac{1}{n}}, \) \(\lim_{n \to \infty} a_n = \frac{1}{3}.\)

2. Write the first four terms of the series \(\sum_{n=3}^{14} \frac{(n-2)!}{(2n-5)!},\) expressing each term as an irreducible fraction. How many terms does the entire series have?

\[
\frac{(3-2)!}{(2 \cdot 3-5)!} \cdot \frac{(4-2)!}{(2 \cdot 4-5)!} \cdot \frac{(5-2)!}{(2 \cdot 5-5)!} \cdot \frac{(6-2)!}{(2 \cdot 6-5)!}; \text{ or } \frac{1}{1 \cdot 1 \cdot 1 \cdot 1}, \frac{2 \cdot 3!}{5!}; \text{ or } \frac{3 \cdot 4!}{7!}; \text{ or } \frac{4 \cdot 5!}{7 \cdot 6 \cdot 5}; \text{ or } \frac{1}{1 \cdot 1 \cdot 1 \cdot 1}, \frac{1}{3 \cdot 20}, \frac{1}{210}. \]

There are \((14-3) + 1 = 12\) terms in all.

3. Construct, if possible, an infinite series such that:

\(a_n > 0\) for all integers \(n; \sum_{n=1}^{\infty} a_n = 7.\) (If not possible, state why.)

Using the form of the sum of an infinite geometric series, \(\frac{a}{1-r}\), where \(a\) is the first term and \(r\) is the common ratio, we want \(\frac{a}{1-r} = 7.\) Taking \(r = 1/7,\) this means that \(\frac{a}{(\frac{6}{7})} = 7\) or \(a = 6;\) so \(\sum_{n=1}^{\infty} 6 \cdot \left(\frac{1}{7}\right)^{n-1} = 7.\) Infinitely many other choices for \(r\) are possible, but most will not lead to such a neat solution. (The choice \(r=6/7\) leads to \(a=1,\) however.)
4. If \( \sum_{n=1}^{\infty} a_n = 5 \) and \( \sum_{n=1}^{\infty} b_n = 7 \), what can you say about
\[
\sum_{n=1}^{\infty} (3a_n + 5b_n)
\]
By Theorem 2.1, parts 1 & 2, adapted to sequences, \( \sum_{n=1}^{\infty} (3a_n + 5b_n) = 3 \sum_{n=1}^{\infty} a_n + 5 \sum_{n=1}^{\infty} b_n = (3)(5) + (5)(7) = 50 \)

b. \( \lim_{n \to \infty} (3a_n + 5b_n) \)?
Since \( \sum_{n=1}^{\infty} (3a_n + 5b_n) \) converges, \( \lim_{n \to \infty} (3a_n + 5b_n) = 0 \).

5. Show that \( \sum_{n=1}^{\infty} \frac{5n(-1)^n}{n^2 + 1} \) converges by citing an appropriate theorem and showing that this sum satisfies the conditions set forth in that theorem.

Looking at the first few terms, it appears that sequence of absolute values of terms is decreasing and is getting “small,” so we might guess that it has limit equal to 0. The series is an alternating series, so we can show that it converges by showing that the sequence of absolute values of terms is decreasing and has limit equal to 0. We want to thus show that for any positive integer \( n \),
\[
\frac{5n}{n^2 + 1} > \frac{5(n+1)}{(n+1)^2 + 1}
\]
which is equivalent to \( (5n)(n^2 + 1) > (n^2 + 1)(5(n+1)) \), or, expanding,
\[
5n^3 + 10n^2 + 5n + 1 > 5n^3 + 5n^2 + 5n + 1
\]
which is clearly true. It remains to show that \( \lim_{n \to \infty} \frac{5n}{n^2 + 1} = 0 \). Dividing numerator and denominator by \( n \), \( \frac{5}{n} \) and \( \frac{1}{n} \), we see that the limit is 0, and the Alternating Series Test implies that the series converges.

Below is the argument presented by the whole class; it picks up at the point of needing to show that \( \frac{5n}{n^2 + 1} > \frac{5(n+1)}{(n+1)^2 + 1} \).
\[
\frac{5n}{(n+1)^2} > \frac{5(n+1)}{(n+1)^2+1} = \frac{5n+5}{(n+1)^2+1+2n+1}
\]
and claims that to compare the \(n\)th term to the \((n+1)\)st term, one should just note that the quantity added to the denominator, which is \(2n+1\), is increasing faster than the quantity added to the numerator, which is 5. However, the argument doesn’t hold in general: If the quantity added to the denominator were just \(n\), instead of \(2n+1\), the inequality would not hold. But it is the case that the quantity added to the denominator, which is now \(n\), is increasing faster than 5, which is the basis for the whole-class argument.

6. Construct, if possible,

a. a series \(\sum_{n=1}^{\infty} a_n\) such that \(\sum_{n=1}^{\infty} |a_n|\) converges, but \(\sum_{n=1}^{\infty} a_n\) diverges; if not possible, state why it is impossible.

According to Theorem 9.5, page 418, such a series is not possible; absolute convergence implies convergence.

b. a sequence of positive terms \(a_n\) such that the alternating series,
\[
\lim_{n \to \infty} a_n = 0, \text{ but } \sum_{n=1}^{\infty} (-1)^n a_n \text{ diverges; if not possible, state why it is impossible.}
\]

According to the Alternating Series Test, Theorem 9.7, page 420, such a series would converge if it had decreasing terms, so for a construction, we seek a sequence of terms that, although having a limit of 0, is not decreasing. Inspired by the Harmonic series, which diverges, but whose corresponding alternating series converges; we look at the sequence:
\[
\frac{2}{n}, \frac{-1}{3}, \frac{2}{4}, \frac{-1}{5}, \frac{2}{6}, \frac{-1}{7}, \ldots, \frac{2}{n}, \frac{-1}{n}, \ldots
\]

Grouping the terms in pairs, we get a sum just like the (divergent) Harmonic series, but the sequence of terms clearly approaches 0. The idea is to keep the terms heading towards 0, but “bump up” every odd term enough from the even one preceding it to get the Harmonic series embedded in the sequence. So, the series:
\[
2/2 - 1/2 + 2/3 - 1/3 + 2/4 - 1/4 + 2/5 - 1/5 + ... + 2/n - 1/n + ... \text{ is an alternating series whose } \text{nth term approaches 0, but is a divergent series. Go to the document bottom for an expression for the nth tterm of this series.}
\]

c. a series \(\sum_{n=1}^{\infty} a_n\) such that \(\sum_{n=1}^{\infty} a_n\) diverges and \(\sum_{n=1}^{\infty} (-1)^n 17a_n\) converges.

(If not possible, state why.)
We know that the Harmonic series diverges and that its corresponding alternating series \( \sum_{n=1}^{\infty} (-1)^n \frac{1}{n} \) converges, as shown in class, by the Alternating Series Test.

The series of constant multiples of the latter series must converge according to Theorem 9.2 (part 1). (Alternatively, one can apply the Alternating Series Test directly to this series of multiples.)

7. Express the repeating decimal 0.132132132... as a geometric series and use that series to express the repeating decimal as a ratio of integers.

\[
0.132132132... = \frac{132}{10^1} + \frac{132}{10^2} + \frac{132}{10^3} + \cdots = \frac{132}{10^1} \left( \frac{1}{1 - \frac{1}{10}} \right) = \frac{132}{999}
\]

8.

a. If the nth term of a series is \( \frac{n}{2n+1} \), does the series converge or diverge; if it converges, what does it converge to; if it diverges, explain why. The limit of the nth term, as n approaches infinity, is 1/2, hence the series must diverge, according to Theorem 9.2 (part 3).

b. If the nth partial sum of a series is \( \frac{n}{2n+1} \), does the series converge or diverge; if it converges, what does it converge to; if it diverges, explain why. The limit of the nth partial sum, as n approaches infinity, is 1/2, hence the series converges to 1/2, according to the definition of convergence of series.

c. Determine an infinite series \( \sum_{n=1}^{\infty} a_n \) whose nth partial sum is \( \frac{n}{2n+1} \), the partial sum in part b) above.

Since \( S_n \) is the sum of the first n terms of the series (\( S_n \)'s last term is \( a_n \)), and \( S_{n-1} \) is the sum of the first n-1 terms, the nth term, \( a_n \), is equal to the difference between the nth and (n-1)st partial sums; i.e., \( a_n = S_n - S_{n-1} = \frac{n}{2n+1} - \frac{n-1}{2n+1} = \frac{2n^2 - n - (2n^2 - n - 1)}{(2n+1)(2n-1)} = \frac{1}{(2n+1)(2n-1)} \)

9.
a. Show that \( \sum_{n=1}^{\infty} \frac{2n+1}{(n+1)(2^n)} \) converges by using the ratio test (or some other test, if you prefer).

The ratio of the \((n+1)\)st term to the \(n\)th term is

\[
\frac{2(n+1)+1}{(n+1)(2^{n+1})} \div \frac{2n+1}{(n+1)(2^n)} = \frac{(2n + 3)(n+1)}{(2n+1)(n+2)(2n+1)} = \frac{2 + \frac{3}{n}(1 + \frac{1}{n})}{1 + \frac{2}{n}(2 + \frac{1}{n})},
\]

so the limit of the ratio of the \((n+1)\)st term to the \(n\)th term is

\[
\lim_{n \to \infty} \frac{2 + \frac{3}{n}(1 + \frac{1}{n})}{1 + \frac{2}{n}(2 + \frac{1}{n})} = 1/2.
\]

By the Ratio Test, since this limit is less than 1, the series converges. You could also show convergence by comparing it to the geometric series of powers of 1/2, showing that

\[
\frac{2n+1}{(n+1)(2^n)} < \frac{1}{(2^{n-1})}.
\]

b. Determine whether \( \sum_{n=3}^{\infty} \frac{(n-2)!}{(2n-5)!} \) converges by applying an appropriate test.

Looking at the ratio of the \((n+1)\)st term to the \(n\)th term,

\[
\frac{(n+1-2)!}{(2(n+1)-5)!} \div \frac{(n-2)!}{(2n-5)!} = \frac{1}{(n-1)! (2n-3)!} = \frac{1}{(n-1)! (2n-3)!} = \frac{1}{(2n-3)(2n-4)} = \frac{(1- \frac{1}{n})}{(2 - \frac{3}{n})(2n-4)},
\]

which approaches 0 as \(n\) approaches infinity, we get by the Ratio Test that the series converges.

Problem #6b: If we use the greatest integer function, \([x]\), which means the greatest integer that is less than or equal to \(x\), this alternating series whose \(n\)th term approaches zero, may be written as:

\[
\sum_{n=1}^{\infty} \frac{2((-1)^{n+1} + 1) + ((-1)^{n+1} - 1)}{2\left[\left\lfloor \frac{n+3}{2} \right\rfloor \right]}.
\]
The greatest integer function, in Excel and TI notation, is denoted by Int(x), a
more natural notation, suggesting the 'integer part' of x. For example, Int(2.3)=2,
Int(2.9)=2, Int(2.99)=2, and Int(3)=3. In the mathematical software package
Maple, the function is called the 'floor' function, floor(x). In this problem, the
greatest integer function serves the purpose of repeating denominators.